Chiral Symmetry Breaking on the Lattice

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August 11, 2017

Abstract

We review important aspects and recent results of chiral symmetry breaking and its restoration on the lattice and mainly focus on the mechanism and origin of these non-perturbative effects.

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1 Introduction

The Standard Model of particle physics is formulated as a quantum theory of gauge fields, describing weak and electromagnetic interactions by electroweak theory and strong interactions by Quantum Chromodynamics (QCD), the theory of quarks and gluons. Quantum field theories have a very well developed perturbation theory for weak couplings. Processes of strongly interacting particles at high momentum transfers, i.e., a large energy scale, are characterized by asymptotic freedom, by decreasing force between color charges with increasing energy. An enormous number of measurements available at high energies can be evaluated within perturbative QCD. In the perturbative regime field theories assume a field-particle duality, associating each field to an elementary particle.

In the hadronic phase quarks and gluons, the elementary fields of QCD, violate field-particle duality and may not serve as asymptotic states in a scattering theory. They have never been detected outside of hadrons and are expected to be confined, i.e., color charged particles (such as quarks) or colorful states cannot be isolated singularly, and therefore cannot be directly observed. There are excellent reviews on the confinement problem and the confinement mechanism, see e.g., \cite{1, 2}. Confinement is a non-perturbative phenomenon just as chiral symmetry breaking (\(\chi_{SB}\)).

According to the quark model mesons consist of quark-antiquark pairs and nucleons of three quarks. Therefore, one would expect that the masses of the lowest-lying mesons are about 2/3 of the nucleon mass of 940 MeV. For the lightest vector meson, the \(\rho\) meson, this is not such a bad approximation. Within the framework of Quantum Chromodynamics (QCD), up- and down-quarks with masses of 3-10 MeV are surrounded by gluon fields, generating an average constituent quark mass of 940/3 MeV \(\approx\) 310 MeV. The dominant part of the nucleon mass is therefore due to the strong interaction between quarks and gluons. The case of the lightest pseudoscalar meson, the pion with a mass of only 15% of the nucleon mass, indicates that QCD is more complicated than just generating constituent quark masses.

Already in the year 1960 Nambu \cite{3} concluded from the low value of the pion mass that the pion is a collective excitation, a Nambu-Goldstone boson, of a spontaneously broken symmetry. He suggested that \(\chi_{SB}\) gives origin to a pseudoscalar zero-mass state, an idealized pion. He formulated an analogy to the BCS-theory of superconductivity, the Nambu-Jona-Lasinio model \cite{4} with chiral symmetry explicitly broken by a term in the Lagrangian. This spontaneous \(\chi_{SB}\) is an effect which is strongly related to the structure of the non-perturbative vacuum of QCD and still elusive to analytical treatment. A very efficient method to tackle this non-perturbative problem is lattice QCD.

In Sect. 2 we formulate QCD in the continuum and on the lattice and take a look at the fate of its symmetries with an emphasis on chiral symmetry and its breakdown. For more details see \cite{5} and references therein. In the third section we discuss the numerical evidence for \(\chi_{SB}\) inferred from lattice calculations and give a few comments on chiral restoration.

In Sect. 4 we discuss various attempts to analyze the mechanism of \(\chi_{SB}\), starting with its relation to confinement and low-lying Dirac modes. The latter are studied intensively regarding their polarization and dimensionality as well as their effect on hadron spectra. Then we focus on the origin of these near-zero modes, reviewing the instanton/monopole and center vortex pictures. We also look at center domains, the chiral magnetic effect and vacuum alignment and finish with a discussion of the relation between the topological approach to functional methods. Finally, we draw our conclusions in Sect. 5.

Due to space/page limitations we restricted our discussion to the mechanism of \(\chi_{SB}\). Further interesting aspects of chiral symmetry on the lattice and beyond are found in these excellent reviews \cite{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.
2 The QCD-Lagrangian and the fate of its symmetries

Like any relativistic quantum field theory, QCD enjoys Poincaré symmetry including the discrete symmetries charge conjugation $C$, parity $P$ and time reversal $T$, each of which is realized. Apart from these space-time symmetries, it also has internal symmetries. Since QCD is an $SU(3)$ gauge theory, it has local $SU(3)$ gauge or color symmetry.

Due to their spin we can assign a handedness or helicity to fermions. Their fields can be decomposed into left- and right-handed components. The mass of fermions couples these components, it breaks chiral symmetry. For massless quarks the QCD-Lagrangian does not have an interaction term between the two quark chiralities, a coupling term like the Nambu-Jona-Lasinio model, breaking chiral symmetry explicitly. Due to the missing interaction term, left- and right-handed fermions can be transformed independently without modifying the Lagrangian.

Up to mass differences of a few percent hadrons can be grouped in multiplets with the same isospin $I$ reflecting an approximate $SU(2)_I$ isospin symmetry of the Lagrangian. The corresponding isospin transformations act on left- and right-handed fermions simultaneously, i.e., $SU(2)_I = SU(2)_{L=R}$. The $SU(3)_F = SU(3)_{L=R}$ flavor symmetry is violated more strongly as seen in the octet of pseudo-scalar mesons. The axial vector symmetry $SU(N_f)_A = SU(N_f)_{L=R}$ the chiral symmetry, on the other hand, is not manifest in the spectrum at all. The experimental evidence for the absence the $SU(N_f)_A$ symmetry is twofold. Axial transformations mix states with different parity. But in the low-lying hadron spectrum one does not observe the corresponding mass-degenerate parity doublets, states with the same quantum numbers, besides parity. The second indication is the above mentioned appearance of (nearly) massless Goldstone bosons. As soon as the chiral symmetry is dynamically broken at low momenta, then necessarily appear Goldstone bosons. We conclude that in massless QCD chiral symmetry is “spontaneously” or “dynamically” broken, it is realized in the Nambu-Goldstone mode, the Lagrangian is chiral symmetric but the vacuum is not.

Despite the absence of interaction terms between left- and right-handed fermions, quarks and anti-quarks are bound in pions to spin-zero states of negative parity. Besides the very light pions $\pi^+$, $\pi^0$, and $\pi^-$ one observes somewhat heavier pseudo-scalars, the four kaons $K^+, K^-, K^0, K^0$ and the $\eta$-meson. According to Goldstone’s theorem, the number of massless bosons is given by the difference of the number of generators of the full symmetry group $G$ and the subgroup $H$ that remains unbroken. In massless QCD the full chiral symmetry group is

\[
G = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B,
\]

while the unbroken subgroup is the flavor symmetry

\[
H = SU(N_f)_{L=R} \otimes U(1)_B.
\]

Hence, in this case one expects $N_f^2 - 1$ massless Goldstone bosons. For $N_f = 2$ there is the isovector triplet of pions with $m_\pi \approx 140$ MeV indicating that in the groundstate of QCD the axial vector symmetry is broken, while for $N_f = 3$ there are eight Goldstone bosons — the pions, the kaons, and the $\eta$-meson. In nature these particles are not exactly massless, because chiral symmetry is explicitly broken by the quark masses. The masses of the up and down quarks are much smaller than the QCD scale $\Lambda_{\text{QCD}} \approx 300$ MeV ($N_f = 3$), which leads to the very small pion mass. The mass of the strange quark, on the other hand, is of the order of $\Lambda_{\text{QCD}}$, thus leading to larger masses of the kaons and the $\eta$-meson. Still, their masses are small enough to identify these particles as pseudo-Goldstone bosons. In the classical massless theory for $N_f$ massless flavors there would be an independent $U(N_f)_V$ symmetry associated with each chirality which can be combined to vector and axial vector symmetries $U(N_f)_V \times U(N_f)_A$, see Sect. 2.1 Sect. 2.2 explains why this full symmetry does not survive quantization, being broken to the above mentioned $SU(N_f)_V \times SU(N_f)_A \times U(1)_B$. For finite quark masses of these chiral symmetries, only the baryon number symmetry $U(1)_B$ is exact.
Due to the dimensionless coupling constant \( g \) classical QCD is approximately scale invariant, \( x_\mu \rightarrow \lambda x_\mu \), for small quark masses. Like \( U(1)_A \) this classical symmetry is broken by quantum fluctuations, the scale symmetry is anomalous, \([16][17][18][19][20][21][22][23][24][25]\).

2.1 Continuum formulation

We work in four-dimensional Euclidean space-time with coordinates \( x \equiv x_\mu = (\vec{x}, x_4) \), with \( \mu = 1 \ldots 3 \) spatial and one temporal direction \( \mu = 4 \). Gluon fields we describe by Hermitian non-Abelian \( su(N_c) \) vector fields

\[
A_\mu(x) = gA_\mu^a(x)T_a,
\]

where we include in analogy to the lattice formulation the gauge coupling \( g \) in the definition. The index \( a \) of the real-valued field components \( A_\mu^a(x) \) runs over the \( N^2_c - 1 \) gauge field components in the color direction of the Hermitian traceless generators \( T_a \) of the \( su(N_c) \) algebra obeying

\[
[T_a, T_b] = if_{abc}T_c, \quad \text{Tr}_C(T_aT_b) = \frac{1}{2}\delta_{ab}, \quad T_a = T_a^\dagger, \quad \text{Tr}_C T_a = 0,
\]

where \( \text{Tr}_C \) indicates the trace over the color matrices in the fundamental representation. Whereas the gauge field transforms under color transformations \( \Omega(x) \in SU(N_c) \) as a connection

\[
A'_\mu(x) = \Omega(x)(A_\mu(x) - i\partial_\mu)\Omega(x)^\dagger,
\]

the algebra-valued field strength

\[
F_{\mu
\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)] = gF_{\mu
\nu}^aT_a,
\]

transforms as a tensor

\[
F_{\mu
\nu}(x) = \Omega(x)F_{\mu
\nu}(x)\Omega(x)^\dagger
\]

and guarantees the gauge invariance of the Euclidean Yang-Mills action

\[
S_{\text{YM}}[A] = \frac{1}{2g^2} \int d^4x \text{Tr}_C(F_{\mu
\nu}F_{\mu
\nu}) = \frac{1}{4} \int d^4x F_{\mu
\nu}^aF_{\mu
\nu}^a.
\]

Due to the gauge freedom (5) a perturbative vacuum \( F_{\mu
\nu} \equiv 0 \) does not necessarily mean a vanishing vector field \( A_\mu \). By gauge transformations (5) a vector field \( A_\mu \equiv 0 \) may be transformed to \( A'_\mu(x) \neq 0 \). Moreover, gauge functions \( \Omega(x) \) defined on a three-dimensional subspace of \( R^4 \), isomorphic to \( S^3 \), may have a winding number defined by the map \( SU(2) \rightarrow S^3 \)

\[
\Pi_3(S^3) \in \mathcal{Z}.
\]

The QCD-vacua are therefore characterized by an integer winding number. Transitions between neighboring winding numbers contribute to the topological charge

\[
Q[A] = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma}\text{Tr}_C(F_{\mu\nu}F_{\rho\sigma}) \in \mathcal{Z},
\]

of a field configuration. Configurations, spherical symmetric in \( R^4 \), with \( Q = 1 \) and minimal action are instantons.

The gluon field mediates the interaction between \( N_f \) quarks. In the path integral formulation of QCD fermions are represented by Grassmann fields. \( \psi(x) \) and \( \psi^+(x) \) have independent generators of the Grassmann algebra for every \( x \), flavor \( f \), color \( c \) and Dirac component \( i \). The Euclidean
Dirac matrices \( \gamma_\mu \) relate the four Dirac components, e.g., \( \overline{\psi} = \psi^\dagger \gamma_4 \). Gauge transformations act with \( \Omega(x) \in SU(N_c) \) on the color indices \( c \) in the fundamental representations \( \{N_c\} \) and \( \{\overline{N}_c\} \)

\[
\psi(x)' = \Omega(x)\psi(x), \quad \overline{\psi}(x)' = \overline{\psi}(x)\Omega(x)^\dagger.
\]  

(11)

With \( \psi(x) \) and \( \overline{\psi}(x) \) we indicate column and row vectors with \( 4 \times N_c \) components running over the Dirac and color components. In \( \Psi(x) \) and \( \overline{\Psi}(x) \) we include even all flavor components. For massless quarks the fermionic action is defined by

\[
S_F[\Psi, \overline{\Psi}, A] = \sum_f \int d^4x \overline{\psi}_f(x) \gamma_\mu \left[ \partial_\mu + i A_\mu(x) \right] \psi_f(x) = \int d^4x \overline{\Psi}(x) \gamma_\mu \left[ \partial_\mu + i A_\mu(x) \right] \Psi(x),
\]

(12)

which is gauge invariant by construction. We will use Euclidean Dirac matrices which are Hermitian and obey the anti-commutation relations

\[
\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu}, \quad \{ \gamma_\mu, \gamma_5 \} = 0, \quad \gamma_5 = \gamma_2 \gamma_3 \gamma_4 \quad \text{with} \quad \gamma_5^\dagger = \gamma_\mu, \quad \gamma_5^\dagger = \gamma_5.
\]

(13)

A convenient choice of the Hermitian matrices is the Weyl- or chiral representation

\[
\gamma = \begin{pmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(14)

where \( \sigma \) are the Pauli matrices. The chiral projectors

\[
P_R = \frac{1 + \gamma_5}{2}, \quad P_L = \frac{1 - \gamma_5}{2} \quad \text{with} \quad P_R P_L = 0, \quad P_R^2 = P_R, \quad P_L^2 = P_L, \quad P_R \gamma_\mu = \gamma_\mu P_L
\]

(15)

map to the eigenvalues \( \pm 1 \) of \( \gamma_5 \). Quark fields can therefore be decomposed into left- and right-handed components

\[
\psi_L(x) = P_L \psi(x), \quad \psi_R(x) = P_R \psi(x), \quad \psi(x) = \psi_L(x) + \psi_R(x),
\]

\[
\overline{\psi}_L(x) = \overline{\psi}(x)P_R, \quad \overline{\psi}_R(x) = \overline{\psi}(x)P_L, \quad \overline{\psi}(x) = \overline{\psi}_L(x) + \overline{\psi}_R(x).
\]

(16)

This decomposition and the properties of the projection operators (15) allow to split the fermionic part (12) of the action in the form

\[
S_F[\Psi, \overline{\Psi}, A] = \int d^4x \left[ \overline{\Psi}_L(x) \gamma_\mu \left( \partial_\mu + i A_\mu \right) \Psi_L(x) + \overline{\Psi}_R(x) \gamma_\mu \left( \partial_\mu + i A_\mu \right) \Psi_R(x) \right].
\]

(17)

This is the important “classical” result: the action for massless fermions decouples into two independent contributions from left- and right-handed quarks. There is no interaction term. As a result, the action of massless QCD is invariant against independent left- and right-handed transformations \( U(N_f)_L \otimes U(N_f)_R \)

\[
\Psi'_L(x) = L \Psi_L(x), \quad \overline{\Psi}'(x) = \overline{\Psi}_L(x)L^\dagger, \quad L \in U(N_f)_L,
\]

\[
\Psi'_R(x) = R \Psi_R(x), \quad \overline{\Psi}'(x) = \overline{\Psi}_R(x)R^\dagger, \quad R \in U(N_f)_R.
\]

(18)

These \( U(N_f)_L \otimes U(N_f)_R \) transformations can be decomposed in vector transformations \( L = R \) and axial vector transformations \( L = R^\dagger \)

\[
U(N_f)_L \otimes U(N_f)_R = SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U(1)_A.
\]

(19)

Quantum effects, the \( U(1)_A \) or axial anomaly, break the \( U(1)_A \) symmetry of the classical action of massless QCD, see Sect. 2.2. From the remaining symmetries, the vector symmetry \( U(1)_B = U(1)_V \) describes the baryon number conservation.
The mass matrix
\[ \mathcal{M} = \text{diag}(m_u, m_d, m_s, \ldots, m_{N_f}) \] (20)
acts on the flavor indices of the fermionic fields and allows to write the mass term of the action in a compact form
\[ S_M[\Psi, \overline{\Psi}] = \int d^4x \left[ \overline{\Psi}_R(x) \mathcal{M}_L(x) \Psi_L(x) + \overline{\Psi}_L(x) \mathcal{M}^\dagger_R \Psi_R(x) \right]. \] (21)
This term couples left- and right-handed fermions and therefore violates \( SU(N_f)_A \), the chiral symmetry. For different quark masses the vector symmetry \( SU(N_f)_V \) breaks down to
\[ SU(N_f) \otimes U(1)_V \rightarrow \prod_{f=1}^{N_f} U(1)_f = U(1)_u \otimes U(1)_d \otimes U(1)_s \otimes \cdots \otimes U(1)_{N_f} \] (22)
and every quark number is separately conserved. The comparison to the experiment shows that the bare quark masses \( m_u \) and \( m_d \) have only a few MeV, much smaller than \( \Lambda_{\overline{MS}} \), and \( m_s \) is of the order \( \Lambda_{\overline{MS}} \). Therefore, \( SU(2)_V \) is broken only slightly and “the eightfold way” \( SU(3)_V \) more strongly. In the limit of massless u-, d- and even s-quarks the total action of QCD
\[ S_{\text{QCD}}[\Psi, \overline{\Psi}, A] = S_{\text{YM}}[A] + S_F[\Psi, \overline{\Psi}, A] + S_M[\Psi, \overline{\Psi}] \] (23)
is symmetric against chiral \( SU(2)_A \) resp. \( SU(3)_A \) transformations. But, in the case of spontaneous breaking of chiral symmetry the ground state of the theory does not respect this symmetry, even for \( S_M[\Psi, \overline{\Psi}] = 0 \).

We should mention that for massless quarks there is only one parameter in the QCD-Lagrangian, the unit \( g \) of the color charge which after renormalization turns out to be a function of the momentum transfer. There is no theoretical prediction yet explaining the values of the quark masses. Only theories beyond the standard model let us hope for an answer to this question. Quantization of the theory with the path integral
\[ Z = \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}A \exp(-S_{\text{QCD}}[\Psi, \overline{\Psi}, A]), \] (24)
produces infinities as long as the theory is not regularized and renormalized. A very successful perturbative regularization is dimensional regularization. Here we focus on the lattice regularization which defines QCD beyond perturbation theory.

Before discussing the lattice formulation we would like to repeat important results concerning the fermionic fields, mainly in the limit of vanishing quark masses. The fermion fields enter the QCD-action in \( S_F \) and \( S_M \) bilinearly. This allows to use the integration formula for Grassmann variables [26]
\[ \int \mathcal{D}[\Psi, \overline{\Psi}] e^{\overline{\Psi}_M \Psi} = \det M, \] (25)
where we used the matrix notation
\[ \overline{\Psi}_M \Psi := \int d^4x d^4y \overline{\Psi}(x) M(x-y) \Psi(y). \] (26)
\( \Psi(x) \) is here a column and \( \overline{\Psi}(x) \) a row vector of Grassmann variables containing all quark components with different flavor, color and Dirac components. Moreover, if no \( (x) \)-dependence is indicated, like in \( \Psi \) and \( \overline{\Psi} \), the vector components run even over all coordinate values. Eq. (25) helps to integrate out fermions in the path integral before the integration over the gluon fields.
2.2 The Axial Anomaly and the Atiyah-Singer index theorem

We now come back to the $U(1)_A$-anomaly, shortly mentioned after Eq. (19), in a fixed gauge background. With the decomposition (17) of the massless fermionic action and the anti-commutation relation (13) one can easily see, that $S_F$ is invariant against the $L = R^T \in U(1)$ axial transformations $U(1)_A$

$$\Psi(x) \to \Psi'(x) = \exp\{i\gamma_5 \theta(x)\}\Psi(x), \quad \bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x) \exp\{i\gamma_5 \theta(x)\}. \quad (27)$$

Surprisingly, as Fujikawa [27] has shown, due to the infinite number of degrees of freedom in the continuum the fermionic path integral measure is not invariant under $\gamma_5$-transformations with an infinitesimal global phase $\theta$ and gives rise to an extra gluon configuration dependent phase factor

$$\int \mathcal{D}[\Psi', \bar{\Psi}'] = \int \mathcal{D}[\Psi, \bar{\Psi}] \exp\{\frac{2i}{32\pi^2} \int d^4x \theta(x) \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu}F_{\rho\sigma})\}. \quad (28)$$

For constant $\theta$ and $N_f$ flavors this results in [27, 6]

$$\int \mathcal{D}[\Psi', \bar{\Psi}'] \int \mathcal{D}[\Psi, \bar{\Psi}] \exp\{2i\theta N_f Q[A]\}. \quad (29)$$

This is the famous $U_A(1)$- or axial anomaly.

For the derivation [27, 6] of Eq. (28) Fujikawa used the massless Dirac operator, see Eq. (12). In our notation with Hermitian Dirac matrices (14)

$$D[A] := \gamma_5(\partial_\mu + iA_\mu), \quad D^\dagger[A] := -D[A], \quad (30)$$

$D[A]$ is anti-Hermitian. For the normalized eigenvectors of $D[A]$ we use the Dirac notation $|\lambda\rangle$ (which may also include a degeneracy of the eigenmodes, especially there may be several zero modes $|0_i\rangle$)

$$D[A]|\lambda\rangle = \lambda|\lambda\rangle, \quad \langle \lambda|\lambda\rangle = 1. \quad (31)$$

Due to the anti-Hermiticity of $D[A]$ the eigenvalues $\lambda$ are purely imaginary

$$\langle \lambda| D[A] |\lambda\rangle = \lambda, \quad \lambda^* = \langle \lambda| D^\dagger[A] |\lambda\rangle = -\lambda. \quad (32)$$

The anti-commutation property

$$\{D[A], \gamma_5\} = 0 \quad (33)$$

implicates that the vectors $\gamma_5|\lambda\rangle$ are eigenvectors to the complex conjugate eigenvalues $\lambda^* = -\lambda$

$$|\lambda\rangle := \gamma_5|\lambda\rangle \quad \text{and} \quad D^\dagger[A] := -D[A] \gamma_5D[A] \gamma_5. \quad (34)$$

Therefore, the eigenvalues appear in complex conjugate pairs $\lambda$ and $-\lambda$ or are zero. Restricted to the space with $\lambda = 0$ we can read Eq. (33) as commutativity of $D[A]$ and $\gamma_5$, $D[A]\gamma_5|0_i\rangle = 0 = \gamma_5D[A]|0_i\rangle$. In this $\lambda = 0$-subspace we can diagonalize $\gamma_5$ and get zero modes of definite chirality

$$\gamma_5|0_j\rangle = \pm|0_j\rangle \quad \Leftrightarrow \quad \langle 0_j| \gamma_5 |0_j\rangle = \pm 1 \quad (35)$$

and their numbers $n_+$ and $n_-$ can be counted. Further we realize: Since eigenvectors to different eigenvalues are orthogonal

$$\langle \lambda| -\lambda\rangle \gamma_5|\lambda\rangle = 0, \quad \lambda \neq -\lambda \quad (36)$$
the complex conjugate pairs do not contribute to the expectation value of the $\gamma_5$ operator. Therefore, only zero modes contribute to this expectation value

$$\sum_\lambda \langle \lambda | \gamma_5 | \lambda \rangle = n_+ - n_-$$  \hspace{1cm} (37)

and therefore the transformation (27) of the path integral measure (29) reads

$$\int D[\Psi', \overline{\Psi}'] \int D[\Psi, \overline{\Psi}] \det(\exp\{-2i\theta \gamma_5\}) = \int D[\Psi, \overline{\Psi}] \prod_\lambda \exp\{-2i\theta \langle \lambda | \gamma_5 | \lambda \rangle\}. \hspace{1cm} (38)$$

The minus sign in the exponent of Eq. (38) takes into account that the measure has to transform inverse to the fields in the integrand. Inserting Eqs. (36) and (35) we get

$$\int D[\Psi', \overline{\Psi}'] \int D[\Psi, \overline{\Psi}] \exp\{-2i\theta (n_+ - n_-)\}. \hspace{1cm} (39)$$

The comparison with the gluonic evaluation (29), presented in Refs. [27, 6], leads to the result

$$N_f Q[A] = n_- - n_+. \hspace{1cm} (40)$$

It relates a fermionic property, the numbers of zero modes to a gluonic quantity, the topological charge $Q[A]$, an integer number counting the vacuum to vacuum transitions of a continuous gauge field, defined in Eq. (10). The Dirac operator is a function of the gauge field and reflects this topological field property in the number of zero modes, as Atiyah and Singer proved [28] in a more general mathematical framework. Eq. (40) results therefore from an application of the Atiyah-Singer index theorem. It allows to determine the “analytical” index

$$Q_f := n_- - n_+ \hspace{1cm} (41)$$

of the Dirac operator via a property of the gauge field and vice versa.

### 2.3 Lattice formulation

There are many good textbooks on the lattice formulation of QCD [29, 30, 31, 32, 26]. We do not repeat the arguments leading to various lattice formulations of the gluonic and the fermionic Lagrangian, for a short overview on lattice fermion formulations see Sect. 2.6. We use standard notations for lattice spacing $a$, spatial and temporal extent $N_s$ and $N_t$ of the lattice, inverse temperature $\beta = aN_t = 1/T$, with the usual choice of natural units $k_B = 1$. The limit $\beta \to \infty$ corresponds to $T \to 0$ and the continuum limit corresponds to $a \to 0$ while keeping $aN_s$ and $aN_t$ fixed. Here, we will concentrate on questions concerning the chiral properties, which are governed by the $\gamma$-matrices, as mentioned in Sect. 2.1. The lattice formulation of the fermionic action suffered for a long time from the fermion doubling problem, from further poles in the propagator around momentum components $\pm \pi/a$. A well-known fermion formulation on a four-dimensional Euclidean hypercubic lattice with sites $x$ and lattice constant $a$, which removes these doubler modes, is an action originally suggested by Wilson [33]

$$D_m(x, y) := m \delta_{x,y} + D_W(x, y),$$

$$D_W(x, y) := 4a \delta_{x,y} - \frac{1}{2a} \sum_{\mu=\pm 1} (1 - \gamma_\mu) U_\mu(x) \delta_{x+\hat{\mu},y} \text{ with } \gamma_- = -\gamma_\mu, U_\mu(x) = U_\mu^T(x - \hat{\mu}), \hspace{1cm} (42)$$
where the vectors $\hat{\mu}$ connect nearest neighbors in $x_\mu$-direction. $U_\mu(x) \in SU(N_c)$ is the parallel transporter from $x$ to $x + \hat{\mu}$. It guarantees gauge invariance. The term proportional to $\gamma_\mu$, the Dirac matrix

$$D_L := \gamma_\mu \nabla_\mu := \frac{1}{2a} \sum_{\mu=1}^{4} \gamma_\mu \left[ U_\mu(x) \left( U_\mu^\dagger(x - \hat{\mu}) \delta_{x+\hat{\mu},y} - U_\mu^\dagger(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right) \right],$$

is the lattice version of the classical Dirac operator $D[A]$ of Eq. (30). The remaining $a$-dependent terms in the Wilson action (42) are proportional to the lattice Laplacian

$$\Delta_L := \sum_{\mu=1}^{4} \Delta_\mu := \sum_{\mu=1}^{4} \frac{1}{a^2} \left[ U_\mu(x) \delta_{x+\hat{\mu},y} - 2 \delta_{x,y} + U_\mu^\dagger(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right]$$

and to the lattice constant $a$ and vanish therefore in the limit $a \to 0$. This momentum dependent $\sim \frac{a}{2} \Delta_L$ term in Eq. (42), the Wilson term, increases effectively the mass $m$ of the doublers by $2/a$ for each momentum component around $p_\mu = \pi/a$, suppressing therefore the influence of the the doublers. But, like the $m$-dependent mass term, the Wilson term contains no factor $\gamma_\mu$ and therefore violates chiral symmetry; a violation which vanishes in the continuum limit only. The fermion doubling problem inhibited for a long time the treatment of dynamical fermions on the lattice. The problem was condensed in a No-go theorem by Nielsen and Ninomiya [34]. Since the solution of the doubling problem is tightly connected to the anomaly and to chiral symmetry breaking we are now going to discuss in more detail, how fermions can be formulated on the lattice with continuous chiral symmetry and without species doubling.

As mentioned in the paragraph before Eq. (11) we formulate fermions with elements of a complex Grassmann algebra for every $x$, flavor $f$, color $c$ and Dirac component $i$. With $\psi(x)$ we indicate a column vector with the $4 \times N_f \times V_c$ components of a single flavor. In $\Psi(x)$ the $N_f$ flavor components and in the $4 \times N_c \times N_f \times V$-dimensional column $\Psi$ the $V_L$ sites of the Grassmann fields are included.

The Dirac matrix $D_L$ of Eq. (43) anti-commutes with $\gamma_5$

$$\{D_L, \gamma_5\} \cong 0.$$  \hspace{1cm} (45)

Therefore, it has the same classical symmetry as the continuum massless Dirac operator $D[A]$ of Eq. (30), a symmetry not respected by quantum theory due to the infinite number of degrees of freedom in the continuum, as we discussed in Sect. 2.2.

On the lattice the number of degrees of freedom is finite and therefore the fermionic measure on the lattice is obviously invariant under a global chiral transformation (27). Since the lattice should reproduce up to order $a$ effects the same spectrum as the continuum theory there must be a mechanism in the lattice formulation breaking the $U_A(1)$-symmetry. A fermion formulation on the lattice with the requested property was suggested by Neuberger [35, 36, 37], see also Sect. 2.6. For his “overlap” fermions the chiral transformation (27) is modified to [38]

$$\Psi' = \exp\{i \theta \gamma_5 (1 - \frac{a}{2} D)\} \Psi, \quad \overline{\Psi}' = \overline{\Psi} \exp\{i \theta (1 - \frac{a}{2} D) \gamma_5\},$$

where $a$ is the lattice constant and $D$ is an appropriately chosen modification of the naive lattice Dirac matrix $D_L$. Looking carefully at the requested invariance

$$\overline{\Psi}' D \Psi' \cong \overline{\Psi}' \exp\{i \theta (1 - \frac{a}{2} D) \gamma_5\} D \exp\{i \theta \gamma_5 (1 - \frac{a}{2} D)\} \Psi \cong \overline{\Psi} D \Psi.$$  \hspace{1cm} (47)

we realize that the invariance (45) of $D_L$ has to be relaxed to

$$(1 - \frac{a}{2} D) \gamma_5 D = D \gamma_5 (1 + \frac{a}{2} D) \Leftrightarrow D \gamma_5 + \gamma_5 D = a D \gamma_5 D.$$  \hspace{1cm} (48)
This invariance can be applied to every factor in the power series of the exponential in Eq. (46), 
\[(1 - \frac{a}{2}D)\gamma_5^n D = D[\gamma_5(-1 + \frac{a}{2}D)]^n,\]
and warrants the invariance (47) of the Lagrangian. Eq. (48) is the celebrated Ginsparg–Wilson relation [39] which had remained unnoticed [40] for a long time. The term \(aD\gamma_5 D\) in Eq. (48) breaks the \(\gamma_5\)-symmetry (45) explicitly, \(D\gamma_5 + \gamma_5 D \neq 0\). It is a term of the order \(a\) which vanishes in the continuum limit. This leads to the interesting interpretation that the fermion doubling problem and the famous Nielsen-Ninomiya No-go theorem [34] – there are no lattice fermions without species doubling and with continuous chiral symmetry – is a manifestation of the anomaly in the flavor singlet axial current [41].

We want to emphasize that, as shown by Lüscher in Ref. [38], for actions fulfilling the Ginsparg–Wilson relation [39] the transformation (46) defines the symmetry (47), which is exact at any given lattice spacing \(a\). For \(a \to 0\) this symmetry converges to the continuum chiral symmetry.

There is another request, the Dirac matrix \(D\) has to fulfill, \(\gamma_5\)-Hermiticity (51), which we are now going to discuss. \(D\) describes the interaction between gluon fields \(U\) and quark fields \(\Psi\). \(D\) is a functional of the gauge field \(D[U]\):

\[D := D[U].\]  

To simplify the notation further on we do not write this functional dependence explicitly. In the path integral over the Grassmann valued fermion fields \(\Psi\) for given gauge configuration \([U]\) and mass matrix \(\mathcal{M}\), see Eq. (20), we can use the bilinearity of the fermionic action in \(\Psi\) and Eq. (26) and generate the fermionic determinant \(\det(D + \mathcal{M})\), a functional of the gauge field. This determinant contains a summation over all possible closed paths of quarks moving under the influence of the gauge field. To describe the reaction of the fermions on the gauge field one tries to take the fermionic determinant as a weight factor in the probability distribution defined by the Euclidean path integral. This is only possible if the fermionic determinant is real and positive

\[\det(D + \mathcal{M}) \overset{!}{=} \left(\det(D + \mathcal{M})\right)^* = \det(D + \mathcal{M})^\dagger.\]  

Due to \(\mathcal{M}^\dagger \overset{!}{=} \mathcal{M}\) the postulate of the reality of the Dirac matrix can be respected by the \(\gamma_5\)-Hermiticity of \(D\)

\[\gamma_5 D \gamma_5 = D^\dagger;\]  

a request, analog to the property (34) in the continuum, and fulfilled by almost all fermionic actions. It describes that the chiral partners of fermions with opposite color charge feel the same gauge field. Attributing to pairs of quarks, like \(u\) and \(d\)-quarks, the same bare mass \(m\), the product of the corresponding two fermionic determinants gives the necessary non-negative weight factor.

The conditions (48) and (51) for \(D\), for a “\(\gamma_5\)-Hermitian Ginsparg–Wilson–Dirac operator”, lead to important consequences which we can read from

\[D + D^\dagger \overset{48}{=} a DD^\dagger, \quad D^\dagger + D \overset{48}{=} aD^\dagger D.\]  

From these two equation we read that \(D\) and \(D^\dagger\) are commuting. This is the definition for \(D\) to be a normal operator. The fermionic matrix \(D\) can therefore be represented by an orthonormal set of eigenvectors \(|\lambda\rangle\) and their eigenvalues \(\lambda\)

\[D = \langle \lambda | \lambda \rangle \quad \Leftrightarrow \quad D |\lambda\rangle = \lambda |\lambda\rangle, \quad \langle \lambda | \lambda \rangle = 1.\]  

We would like to emphasize that \(|\lambda\rangle\) are row vectors and \(\bar{\Psi}_\lambda\) column vectors of complex numbers for every lattice site \(x\), flavor \(f\), color \(c\) and Dirac component \(i\). The original Grassmann variables of the fermionic fields get integrated out in the fermionic path integral [25]. After performing this integration we can work with matrices like \(D\) and their determinants.
From Eq. (51) follows that $\gamma_5 |\lambda \rangle$ is an eigenvector of $D$ to the eigenvalue $\lambda^*$

$$D \gamma_5 |\lambda \rangle = \lambda^* |\lambda \rangle,$$  

(54)

Thus, the eigenvalues are either real or they appear in complex conjugate pairs $\lambda, \lambda^*$ with eigenfunctions $|\lambda \rangle$ and $|\lambda^* \rangle = \gamma_5 |\lambda \rangle$. From Eq. (52) we get further interesting properties of the eigenvalues: If we insert the two eigenvalue equations (53) and (54) into Eq. (52) we get

$$\lambda + \lambda^* a \lambda \lambda^*.$$  

(55)

The polar representation $\lambda = |\lambda| e^{i\alpha}$ shows a nice application of Thales' theorem

$$\cos \alpha = \frac{a |\lambda|}{2}$$  

(56)

the “Ginsparg-Wilson circle” of eigenvalues, see Fig. 1, where $|\lambda|$ is the hypotenuse and $2/a$ is the adjacent of $\alpha$ in a right-angled triangle. With $1/\lambda = |\lambda|^{-1} e^{-i\alpha}$ we get simple expressions for the real and imaginary parts for $1/\lambda$ which we will use in Eq. (82) in Sect. 2.5

$$\frac{1}{\lambda} a \frac{e^{-i\alpha(\lambda)}}{2 \cos \alpha(\lambda)} = \frac{a}{2} - i \frac{a}{2} \tan \alpha(\lambda) = \frac{a}{2} + i \frac{a}{2} \tan \left(\frac{\phi(\lambda)}{2}\right),$$  

since $\alpha + \frac{\phi}{2} = \frac{\pi}{2}.$

The eigenvectors $|\lambda \rangle$ and $|\lambda^* \rangle$ are orthogonal since they belong to different eigenvalues $\lambda \neq \lambda^*$

$$\langle \lambda | \gamma_5 |\lambda \rangle = 0 \quad \text{for} \quad \lambda \neq \lambda^*.$$  

(58)

This equation tells us also that the expectation value of $\gamma_5$ in the state $|\lambda \rangle$ is vanishing, if $\lambda$ is not real.

The eigenvectors to the real eigenvalues $\lambda$ are not paired. Further, they have the important property of good chirality, as can be easily seen: From the right equation in (54) we read for real $\lambda = \lambda_r \in \{0, \frac{2}{a}\}$

$$D \gamma_5 |\lambda_r \rangle = \gamma_5 D |\lambda_r \rangle,$$  

(59)

that $D$ and $\gamma_5$ are commuting in the subspace of $\lambda_r$ and therefore simultaneously diagonalizable. The eigenfunctions $|\lambda_r \rangle$ can be chosen with good chirality, they are “chiral”,

$$\gamma_5 |\lambda_r \rangle = \pm |\lambda_r \rangle.$$  

(60)
Of special importance are the chiralities of the zero modes, of the modes with \( \lambda = 0 \). With \( n_+ \) we count the number of right-handed or positive chirality modes and with \( n_- \) the left-handed modes with negative chirality.

\( \gamma_5 \) is diagonal in the Weyl basis \([14]\) with \( \text{Tr}_D \gamma_5 = 0 \) in each subspace of four free Dirac spinors with the same momentum. Due to the invariance of the trace under basis transformations, it follows that also in the full basis of eigenstates of \( D \) we get

\[
\sum_{\lambda} \langle \lambda | \gamma_5 | \lambda \rangle = 0. \tag{61}
\]

In lattice simulations one of the numbers \( n_+ \) and \( n_- \) is always zero for usual anti-periodic boundary conditions. All zero modes have the same chirality. According to the vanishing trace (61) each zero mode with given chirality needs a chiral partner with opposite chirality. Such states are available only at \( \lambda = \frac{2}{a} \). In the continuum limit \( a \to 0 \) these “doubler modes” are sent to infinite eigenvalues.

Due to the orthogonality \([58]\) only zero- and doubler-modes contribute to Eq. (61), with \( \pm 1 \). We remove also the contributions of the doublers by multiplying the summands in Eq. (61) with the factor \( a/2 - 1 \) and get \( (a\lambda/2 - 1) \langle \lambda | \gamma_5 | \lambda \rangle = \langle \lambda | (aD/2 - 1) \gamma_5 | \lambda \rangle \). Now only the zero modes contribute, with \( -\langle \lambda | \gamma_5 | \lambda \rangle \). This allows to count the difference between the zero modes of different chirality, the “analytical” index \([41]\)

\[ Q_f := \sum_{\lambda} \langle \lambda | (a/2 - 1) \gamma_5 | \lambda \rangle = n_- - n_+. \tag{62} \]

This index has also appeared in the fermionic measure, in the axial anomaly in the continuum, in Sect. 2.2.

### 2.4 The Axial Anomaly on the lattice

On the lattice the measure is invariant under the modified chiral transformation \([46]\), due to the finite number of integrations. But the result of the Grassmann integrations \([25]\), the determinant of the fermionic matrix \( D \) is not invariant. We get from Eq. (47)

\[
\det D' := \det \left[ \exp \left\{ i \theta \left( 1 - \frac{a}{2} D \right) \gamma_5 \right\} \right] \det D \det \left[ \exp \left\{ i \theta \gamma_5 \left( 1 - \frac{a}{2} D \right) \right\} \right]. \tag{63}
\]

As we will see immediately below, the determinant is modified by the square of the determinant of the transformation matrix \( T \). In the eigenbasis \( |\lambda\rangle \) of \( D \) one can easily see that the determinant of \( T \) gets the exponential of the logarithm of the trace

\[
(\det T)^2 = e^{2\text{Tr} \ln T}, \quad T := \exp \left\{ i \theta \left( 1 - \frac{a}{2} D \right) \gamma_5 \right\}. \tag{64}
\]

With \( \text{Tr} \ln T = -i \theta \text{Tr} \left[ \left( \frac{a}{2} D - 1 \right) \gamma_5 \right] = -i \theta \text{Tr} \gamma_5 \left( \frac{a}{2} D - 1 \right) \) we can immediately apply Eq. (62) and get

\[
\det D' \overset{63}{=} \exp \{-2i\theta N_f Q_f\} \det D, \quad \text{with} \quad Q_f \overset{62}{=} n_- - n_+. \tag{65}
\]

Up to the sign in the phase this is the same result as Fujikawa got in the continuum. The different sign reflects the fact that on the lattice we transform the fermionic matrix \( D \) \([63]\) which transforms inverse to the fermionic measure \([39]\). The comparison to the gluonic evaluation, which is not done here, leads again to a relation between the analytical index \( Q_f \), a property of the Dirac matrix, and a gluonic property, the topological charge \( Q[A] \)

\[
Q_f \overset{62}{=} n_- - n_+ = N_f Q[A]. \tag{66}
\]
This relation between the analytical and the topological index was announced in 1963 by Michael Atiyah and Isadore Singer for elliptic differential operators on compact manifolds \[^{42}\]. They published various generalizations in a sequence of papers from 1968 to 1971 \[^{28}\]. On the lattice, the theorem applies to any action that satisfies the Ginsparg-Wilson relation, including the Neuberger overlap action, however, it does not necessarily hold for non-Ginsparg-Wilson actions.

### 2.5 Chiral condensate on the lattice

The chiral \( U(1)_A \)-transformation \[^{46}\] acts on all fermions symmetrically. In Sect. \[^{2.4}\] it turned out that this symmetry is anomalous, it is broken by the fermionic integration. This anomaly has an experimental consequence, it prevents the \( \eta' \)-meson with 957.8 MeV to be a Goldstone boson. Pions, with 135.0 and 139.6 MeV are much lighter than expected from their quark content. Also Kaons with 493.7 and 497.6 MeV and the \( \eta \)-meson with 547.9 MeV have a mass smaller than two thirds of the mass of baryons in the lowest octet. We can therefore expect that the chiral limit with \( N_f = 2 \) is a very good approximation and \( N_f = 3 \) is still good. In the approximation of massless quarks we can generalize \( U(1)_A \) to \( U(N_f)_A \). The additional global \( SU(N_f)_A \)-transformations

\[
\Psi' = \exp \{ i \theta_a T_a (1 - \frac{a}{2} D) \} \Psi, \quad \overline{\Psi}' = \overline{\Psi} \exp \{ i \theta_a T_a (1 - \frac{a}{2} D) \gamma_5 \},
\]

(67)
do not lead to new anomalies since the \( SU(N_f)_A \)-generators \( T_a \) are traceless. For a proof we evaluate in analogy to the discussion in Sect. \[^{2.4}\] the square of the determinant of the transformation matrix \( T(T_a) \)

\[
[\det T(T_a)]^2 := \det^2 \left[ \exp \left\{ i \theta_a T_a (1 - \frac{a}{2} D) \gamma_5 \right\} \right],
\]

(68)
use again Eq. \[^{64}\]

\[
[\det T(T_a)]^2 \overset{\[^{64}\]}{=} \exp \left\{ -2i \theta_a \text{Tr} \left[ T_a \left( 1 - \frac{a}{2} D \right) \gamma_5 \right] \right\} \overset{\[^{64}\]}{=} 1.
\]

(69)
and get no contribution from the determinant due to the assumed flavor symmetry of the fermionic matrix \( D \). To receive this result we perform in the trace \( \text{Tr} \), running over lattice sites \( x \), color \( c \), Dirac \( i \) and flavor \( f \) indices, the flavor trace first and use the vanishing trace of the \( SU(N_f)_A \)-generators.

Even for vanishing quark masses the chiral vector symmetry \[^{67}\] may be broken, left- and right-handed fermions may be coupled by the dynamics of QCD. To adjust the fermion formulation in analogy to the continuum to the modified chiral transformation \[^{67}\] we have to decompose the mass term \[^{21}\] and the kinetic term \[^{17}\] of the Lagrangian on the lattice using the Ginsparg-Wilson relation \[^{48}\]. The decomposition of the mass term seems to go as usual. For the kinetic term we have to use a modification of the projection operators \[^{15}\]. Very helpful for this aim is the relation

\[
\gamma_5 (1 - aD) \gamma_5 (1 - aD) = 1 - a \{ D + D^\dagger - aD^\dagger D \} = 1 \quad \Leftrightarrow \quad \gamma_5^2 = 1 \text{ with } \gamma_5 := \gamma_5 (1 - aD),
\]

(70)
which allows to define the modified projection operators \[^{15}\]

\[
\hat{P}_R := \frac{1 + \gamma_5}{2}, \quad \hat{P}_L := \frac{1 - \gamma_5}{2} \quad \text{with} \quad \hat{P}_R^2 = \hat{P}_R, \quad \hat{P}_L^2 = \hat{P}_L, \quad \hat{P}_R \hat{P}_L = 0, \quad \hat{P}_R + \hat{P}_L = 1.
\]

(71)
For the decomposition of the fermionic matrix \( D \) we use another way of writing the Ginsparg-Wilson relation \[^{48}\] and get the asymmetric relations

\[
D \hat{P}_R = (D + D\gamma_5 - aD\gamma_5 D)/2 = \hat{P}_L D, \quad D \hat{P}_L = (D - D\gamma_5 - aD\gamma_5 D)/2 = \hat{P}_R D.
\]

(72)
We apply therefore different projectors for $\Psi$ and $\overline{\Psi}$

\[ \overline{\Psi}_R := \overline{\Psi} P_L, \quad \overline{\Psi}_L := \overline{\Psi} P_R, \quad \Psi_R := \hat{P}_R \Psi, \quad \Psi_L := \hat{P}_L \Psi \]  

(73)

and arrive at a decomposition of the fermionic matrix in analogy to the continuum

\[ \overline{\Psi} D \Psi = \overline{\Psi}_R D \Psi_R + \overline{\Psi}_L D \Psi_L. \]  

(74)

The asymmetric definition $\Psi$ of the projected wave functions has consequences for the mass term (21). If we define this term on the lattice with the projected components

\[ \overline{\Psi}_L \mathcal{M} \Psi_R + \overline{\Psi}_R \mathcal{M} \Psi_L \]  

(73)

\[ \overline{\Psi} \mathcal{M} (\frac{P_R \hat{P}_R}{P_R (1 - \frac{a}{2} D)} + \frac{P_L \hat{P}_L}{P_L (1 - \frac{a}{2} D)}) \Psi = \overline{\Psi} \mathcal{M} (1 - \frac{a}{2} D) \Psi \]  

(75)

we get an additional term leading via the fermionic matrix $D$ to an additional coupling of neighboring lattice sites. We combine this mass term with the kinetic term and get the total fermionic action of Ginsparg-Wilson fermions

\[ S_{GW} := \overline{\Psi} \left(1 - \frac{a \mathcal{M}}{2}\right) D \Psi + \overline{\Psi} \mathcal{M} \Psi = \overline{\Psi} D_{\mathcal{M}} \Psi \]  

mit $D_{\mathcal{M}} := D + \mathcal{M} \left(1 - \frac{a}{2} D\right)$.  

(76)

As mentioned before, $\Psi$ is a column vector and $\overline{\Psi}$ a row vector containing Grassmann variables for every lattice site $x$, flavor $f$, color $c$ and Dirac component $i$. For the further calculations it is important that the mass matrix $\mathcal{M}$ is block diagonal in the flavor quantum number $f$. Below, we need especially the submatrix $D^\mu$ for the lightest quark flavor $u$ with mass $m$ and the submatrix

\[ D_m := \begin{bmatrix} D^\mu + m \left(1 - \frac{a}{2} D^\mu\right) \end{bmatrix}. \]  

(77)

of $D_{\mathcal{M}}$.

If chiral symmetry would be intact left- and right-handed quarks could be transformed independently and there would be no coupling between left- and right-handed quarks, a coupling as it appears in the mass term (75). If the invariance is lost, we call the symmetry broken and define an order parameter for the breaking of chiral symmetry, the “quark condensate” $\Sigma$. For its definition (86) we use the scalar expectation value

\[ \Sigma(a, m, V_L) := \frac{1}{a^4 V_L} \left\langle \bar{u} \left(1 - \frac{a}{2} D^\mu\right) u \right\rangle \]  

(78)

of the bilinear $\bar{u}(x) u(x)$ with a form indicated by the mass term (75) for the lightest Grassmann valued quark field $u(x)$ in the infinite volume $V = a^4 V_L$, vanishing $u$-quark mass $m$ and continuum $a \rightarrow 0$ limit. The fermionic bilinear $\bar{u} \left(1 - \frac{a}{2} D^\mu\right) u$ includes besides sums over Dirac indices $i$ and color indices $c$ also a sum over lattice sites $x$, therefore we divide in Eq. (78) by the physical volume $V$. The brackets $\langle \rangle$ in Eq. (78) indicate the gluonic and fermionic path integrals. According to Eq. (25) the Grassmann integration over the fermionic bilinear leads to a subdeterminant, where one row and one column, corresponding to the components of $u(x)$ and $\bar{u}(x)$, of the fermionic matrix are removed. Consequently, the subdeterminants for flavors $f \neq u$ are unaffected and only the subdeterminant of the $u$-quarks has to be treated in detail. From linear algebra we know that a matrix element of the inverse matrix is just such a subdeterminant divided by the determinant. Taking all this into account we get

\[ \Sigma(a, m, V_L) \frac{1}{a^4 V_L} \left\langle \text{Tr} \left[ \left(1 - \frac{a}{2} D^\mu\right) D_m^{-1} \right] \right\rangle_G. \]  

(79)
The sign change from Eq. (78) to Eq. (79) originates in the sign of the exponent of the fermionic Boltzmann factor \( \exp\{-S_{GW}\} \). The matrices \( D^\mu \) and \( D_m \) and the trace \( \text{Tr} \) in Eq. (79) extend only over the \( u\)-quark degrees of freedom and the full determinant of \( D_{\mathcal{M}} \) is included in the gluonic path integral

\[
\langle O \rangle_G := \frac{1}{Z} \int \mathcal{D}U e^{-S_\chi[U]} O[U] \det(D_{\mathcal{M}}[U]).
\]  

(80)

The study of the dependence of \( \Sigma(a,m,V_L) \) on the volume \( V \), the quark mass \( m \) and the lattice constant \( a \) will be discussed in Sect. 3.1. Here we describe the analytic evaluation of the path integral (78) for given \( a \) in the \( V \to \infty \) and \( m \to 0 \), the “chiral” limit.

In the \( m \to 0 \) limit of Eq. (78) we find a nice interpretation of the factor \( 1 - \frac{q}{2} D \) characteristic for Ginsparg-Wilson fermions, which appeared in the chiral transformation (67) and in the mass integral (75) for Ginsparg-Wilson fermions, which appeared in the chiral transformation (67) and in the mass integral (75)

\[
\lim_{m \to 0} \text{Tr} \left[ \left( 1 - \frac{a}{2} D^\mu \right) D^{-1}_m \right] = \frac{1}{2} \text{Tr} \left( \frac{1 - \frac{q}{2} D^\mu}{m(1 - \frac{q}{2} D^\mu) + D^\mu} \right) = \frac{1}{2} \sum_{\lambda} \frac{1}{\lambda^2 - \frac{a^2}{4}}.
\]  

(81)

We recognize that in the \( m \to 0 \) limit the factor \( 1 - \frac{q}{2} D^\mu \) removes the real part of the eigenvalues \( 1/\lambda \) of \( 1/D^\mu \). Further we realize that for the evaluation of the trace in Eqs. (79) and (81) we should treat the eigenvectors for \( \lambda = 0 \) separately. For the “operator” in expression (79) we get

\[
\text{Tr} \left[ \left( 1 - \frac{a}{2} D^\mu \right) D^{-1}_m \right] = \text{Tr} \left( \frac{1 - \frac{q}{2} D^\mu}{m(1 - \frac{q}{2} D^\mu) + D^\mu} \right) = \frac{1}{2} \sum_{\lambda} \frac{1}{\lambda^2 - \frac{a^2}{4}}.
\]  

(82)

where \( n_+ \) and \( n_- \) are the numbers of right-handed and left-handed zero modes and \( \phi_\lambda \) is the angle \( \phi \) of Fig. 1 related to the complex eigenvalue \( \lambda \) on the Ginsparg-Wilson circle with positive or negative imaginary part. Extending in Eq. (82) the summation over all non-real eigenvalues \( \lambda \neq \lambda^* \) we want to indicate also that the doublers with \( \lambda = 2/a \) do not contribute to the trace. Due to the factor \( 1/V \) in Eq. (79) also the zero modes do not contribute in the \( V \to \infty \) limit since their number \( n_+ + n_- \) does not grow faster than \( \sqrt{V} \).

For continuous

\[
y := \frac{a}{2} \tan \frac{\phi_\lambda}{2} \approx \frac{\phi_\lambda}{a} \approx -i\lambda = \pm |\lambda| \quad \text{for} \quad \phi \approx 0
\]  

(83)

the expression in the square bracket of Eq. (82) approaches in the \( m \to 0 \) limit a \( \delta \)-function at \( \lambda = 0 \)

\[
\lim_{m \to 0} \frac{1}{2} \left[ \frac{1}{m + iy} + \frac{1}{m - iy} \right] = \pi \delta(y), \quad \int \frac{m}{m^2 + y^2} \, dy = \int \frac{\, dy}{1 + \frac{y^2}{m^2}} = \frac{\pi}{m} \quad \text{for} \quad \phi_\lambda \in [0, \pi].
\]  

(84)

Approaching the sum over \( \lambda \) in Eq. (82) by an integral over a density function \( \rho_\lambda \) we can formally perform an integral over the delta-function. Due to the distribution of eigenvalues on the Ginsparg-Wilson circle we expect for given \( m \) a density function \( \rho_\lambda(\phi,m) \) to be approximately independent of \( a \) and the number of eigenvalues to increase with \( V_L \). Including the gluonic average and the remaining factor \( a^{-4} \) of Eq. (79) we define therefore

\[
\frac{1}{a^4} \int \frac{1}{V_L} \sum_{|\lambda| < \Lambda} \langle G \rangle \to \frac{1}{a^4} \int_{-a\Lambda}^{a\Lambda} d\phi \, \rho_\lambda(\phi,m).
\]  

(85)
Combining Eq. (84) with Eq. (85) we define the chiral condensate

\[ \Sigma := \lim_{a \to 0} \lim_{m \to 0} \lim_{V_L \to \infty} \Sigma(a, m, V_L) = \lim_{m \to 0} \lim_{a \to 0} \lim_{V_L \to \infty} \pi \rho_\lambda(0, m) := \pi \rho(0). \] (86)

This result is the famous Banks-Casher relation [43].

To break a symmetry spontaneously requires an infinite system, the limit of the number \( V_L \) of lattice points going to infinity. Like in other cases of spontaneous symmetry breaking the \( V \to \infty \), the “thermodynamic” limit, cannot be interchanged with the removal of the explicit symmetry breaking term, in our case the \( m \to 0 \) “chiral” limit. The results of numerical calculations should be independent of the order of the other limits.

In lattice calculations the Banks-Casher relation is an optimal tool to determine the value of the chiral condensate from the density of near-zero modes of the Dirac operator. If this density is non-vanishing then the chiral symmetry is broken. It is interesting to know the value of the condensate for various gluonic coupling constants, quark masses, baryonic densities and temperatures. Some of these calculations are discussed in Sect. 3.1. These calculations do not explain why chiral symmetry can be broken, they do not say anything about the mechanism of chiral symmetry breaking. This mechanism is another interesting question which will be discussed in Sect. 4.

For completeness we should mention that the chiral condensate has not only been obtained using the Banks-Casher relation but also from the Gell-Mann-Oakes-Renner relation and using the Wilson flow as will be discussed briefly in Sect. 3.1 too. Finally, the eigenvalue distribution of the Dirac operator can also be obtained through the application of (chiral) Random Matrix Theory to QCD, where the detailed dependence of the partition function and Dirac operator eigenvalue correlation functions on finite lattice spacing \( a \) or chemical potential \( \mu \) are computed. For an excellent review and recent developments see [44] and references therein.

### 2.6 Overview of lattice fermion formulations

In this section we briefly summarize the fermionic actions used by various groups for the results presented in this review. We already introduced the standard Wilson action [33] in Eq. (42), which successfully removes fermion doublers, but suffers from discretization errors of \( O(a) \) and breaks chiral symmetry explicitly. A systematic improvement program reducing the cut-off errors order by order in \( a \) has been proposed by Symanzik [45] and developed for on-shell quantities in Ref. [46]. Sheikholeslami and Wohlert [47] proposed to add the so-called clover term

\[ \mathcal{L}_1 = - \frac{a}{4} c_{SW}(g) \sum_x \bar{\psi}(x) \sigma_{\mu \nu} F_{\mu \nu}(x) \psi(x), \] (87)

to the original Wilson action and use chiral symmetry to fix the coefficients in \( c_{SW} \). A non-perturbative evaluation of this function leads to [48]

\[ c_{SW}(g) = \frac{1 - 0.656 g^2 - 0.152 g^4 - 0.054 g^6}{1 - 0.922 g^2}, \ g^2 \leq 1. \] (88)

This NP improved Wilson action still shows the problem of the appearance of small eigenvalues of the lattice Dirac operator [46], which can be overcome using the so-called twisted mass formulation of QCD, where the mass term in the Dirac operator is chirally twisted [49]. Dynamical simulations with twisted Wilson fermions allow to approach the physical point at realistic values of quark masses.
In some way, the twisted mass plays a similar role as an infrared cut-off as the quark mass in the staggered fermion or overlap approach does.

**Staggered fermions** result from naive doubled lattice fermions by so-called spin diagonalization \([50, 51]\), which reduces the fermion multiplication factor \(2^d = 16\) to \(16/4 = 4\). Hence, for \(d = 4\), staggered fermions represent 4 flavors of mass-degenerate fermions. They have a single 1-component pair of Grassmann variables \(\chi_x\) and \(\chi_y\) per lattice point \(x\). The corresponding lattice action for free staggered fermions takes the form

\[
S[\chi, \bar{\chi}] = a^4 \sum_{x, \mu} \left( \chi_{x+\mu} \bar{\chi}_{x+\mu} - \chi_{x-\mu} \bar{\chi}_{x-\mu} \right) + a^4 \sum_x m \bar{\chi}_x \chi_x, \tag{89}
\]

where

\[
\eta_{x,1} = 1, \quad \eta_{x,2} = (-1)^{x_1/a}, \quad \eta_{x,3} = (-1)^{(x_1+x_2)/a}, \quad \eta_{x,4} = (-1)^{(x_1+x_2+x_3)/a}. \tag{90}
\]

For \(m = 0\) the staggered fermion action has an exact \(U(1)_e \otimes U(1)_o\) symmetry

\[
\chi'_x = \exp(i \varphi_e) \chi_x, \quad \bar{\chi}'_x = \bar{\chi}_x \exp(-i \varphi_o), \quad \text{for } (x_1 + x_2 + x_3 + x_4)/a \text{ odd,}
\]

\[
\chi'_x = \exp(i \varphi_e) \chi_x, \quad \bar{\chi}'_x = \bar{\chi}_x \exp(-i \varphi_e), \quad \text{for } (x_1 + x_2 + x_3 + x_4)/a \text{ even,}
\]

which is a subgroup of the \(SU(4)_e \otimes SU(4)_o \otimes U(1)_B\) chiral symmetry of the corresponding continuum theory. In the interacting theory the chiral and flavor symmetries besides \(U(1)_e \otimes U(1)_o\) are explicitly broken by the staggered fermion action. These symmetries are recovered only in the continuum limit, but since they have a remnant of chiral symmetry and are relatively easy to simulate numerically, staggered fermions provide a convenient framework for studies of chiral symmetry breaking at \(N_f = 4\).

\(O(a^2)\) tadpole improved or **asqtad staggered fermions** were introduced in \([52, 53]\).

Finally, domain wall and overlap fermions proposed a novel method to preserve chirality on the lattice. The former use the fact that chiral fermions become trapped on domain walls \([54]\). Kaplan \([55]\) used a Wilson-Dirac operator in five dimensions with a mass term that is a function of the fifth direction. In particular, the mass term changes sign in the 5th dimension creating a four-dimensional domain wall at the points where it vanishes. A four-dimensional chiral fermion is then trapped on the domain wall, this fermion is commonly referred to as a domain wall fermion and its action is constructed on a five-dimensional space-time lattice with coordinates \((x, x_5)\), where \(x\) refers to the usual four dimensions and \(x_5 \in \{a_5, 2a_5, ..., L_5\}\) refers to the fifth direction of finite extent \(L_5\). Since the fifth direction is physically different from the other directions a new lattice spacing \(a_5\) is introduced in that direction. The **domain wall fermion action** is given by

\[
S_F[\Psi, \bar{\Psi}, U] = -a^4 a_5 \sum_{x, x_5, y, y_5} \bar{\Psi}_{x, x_5} D_{DW} [U]_{x, x_5, y, y_5} \Psi_{y, y_5} \tag{92}
\]

with the domain wall Dirac operator

\[
D_{DW} [U]_{x, x_5, y, y_5} = \delta_{x_5, y_5} D^\parallel [U]_{x, y} + \delta_{x, y} D^\perp [U]_{x_5, y_5},
\]

\[
D^\parallel [U]_{x, y} = M \delta_{x, y} + \sum_{\mu} \frac{1}{2a} \left( \gamma_\mu U_{x, \mu} \delta_{x+\mu, y} - \gamma_\mu U^\dagger_{x-\mu, y} \delta_{x-\mu, y} \right) - \sum_{\mu} \frac{1}{2a} \left( 2 \delta_{x, y} - U_{x, \mu} \delta_{x+\mu, y} - U^\dagger_{x-\mu, y} \delta_{x-\mu, y} \right),
\]

\[
D^\perp [U]_{x_5, y_5} = \begin{cases} (P_R \delta_{a_5, y_5} - \delta_{a_5, y_5})/a_5 - m P_L \delta_{a_5, y_5} & \text{for } x_5 = 1, \\
(P_R \delta_{x_5-a_5, y_5} + P_L \delta_{x_5, y_5} - \delta_{x_5, y_5})/a_5 & \text{for } a_5 < s < L_5, \\
(P_L \delta_{a_5-a_5, y_5} - \delta_{a_5, y_5})/a_5 - m P_R \delta_{a_5, y_5} & \text{for } x_5 = L_5.
\end{cases} \tag{93}
\]

Here \(P_R\) and \(P_L\) are the chiral projection operators defined in Eq. (15). In order to produce massless quarks one should set \(0 \leq M a_5 \leq 2\) at tree level \([55]\) and take \(L_5 \to \infty\), see also \([56]\).
Narayanan and Neuberger developed an idea of using an infinite number of regulator “flavor” fields to preserve chirality [57], which is basically equivalent to the domain wall approach since the fifth dimension is analogous to a flavor space. Neuberger realized that it is possible to find an analytic formula for an effective Dirac operator that describes the massless chiral mode of the domain wall fermion. The determinant of a chiral fermion in the background of a gauge field is equivalent to the overlap of two many-body fermionic ground states [58]. Using his insight on the overlap formula for vector-like gauge theories [59], he found a simple and elegant formula for the four-dimensional Dirac operator [35], which is referred to as the overlap Dirac operator given by

\[ D_O[U] = \frac{1}{2a} \left[ 1 + \gamma_5 \frac{H[U]}{\sqrt{H[U]^2}} \right], \tag{94} \]

where \( H[U] = \gamma_5 D[|U|] \) and \( D|[U] \) is the operator defined in Eq. (93) where one needs to set \( 0 \leq M a_5 \leq 2 \) as before in order to obtain massless quarks.

It is straightforward to check that \( D_O[U] \) indeed satisfies the Ginsparg-Wilson relation, but it is also possible to show that if one wants to benefit from good chiral properties of Ginsparg-Wilson fermions, one has to give up the notion of ultralocal actions [60]. However, as has been shown in [61], close to the continuum limit the couplings in the overlap Dirac operator fall off exponentially with the distance. In this sense these new Dirac operators are still local.

There have been efforts to generalize the Ginsparg-Wilson relation [62] and use this as a guide to construct new classes of Dirac operators [63]. Since Dirac operators which satisfy the Ginsparg-Wilson relation exactly are computationally very demanding, there have also been efforts to find perfect Dirac operators that satisfy the Ginsparg-Wilson fermions approximately [64]. Another approach has been to expand the most general lattice Dirac operator in a basis of simple operators. The coefficients of the expansion then are determined using the Ginsparg-Wilson relation [65]. This approach has been used to construct a practical operator for lattice simulations [66], the chirally improved (CI) Dirac operator.

Finally, another issue to be discussed with fermions on the lattice is how we can introduce a finite quark number density resp. the quark chemical potential \( \mu \). The latter always enters in the form \( \mu / T \) in partition function of the grand canonical ensemble

\[ Z(T, \mu) = \text{Tr} \left[ e^{-(\hat{H} - \mu \hat{N}_q)/T} \right], \tag{95} \]

multiplying the quark number operator \( \hat{N}_q \). The Euclidean continuum quark number operator is given by the spatial volume integral over the temporal component \( \bar{\psi}(x) \gamma_4 \psi(x) \) of the conserved current \( \bar{\psi} \gamma_\mu \psi \). On the lattice, the quark number is the conserved charge of the \( U(1) \) global symmetry. Determining the Noether current for the lattice action gives the current expressed by nearest neighbor terms. The space integral then produces a suitable form of the chemical potential term. One therefore implements the chemical potential by multiplying the temporal hopping terms in the quark action with \( \exp(a \mu N_t) \). In practice this can be achieved by modifying all forward time-directed hopping terms in a single time slice with the factor \( \exp(a \mu N_t) = \exp(\mu / T) \), and the corresponding backward-oriented terms with the inverse factor. However, the introduction of the chemical potential comes with a serious technical drawback: For \( \mu \neq 0 \) the Dirac operator is no longer \( \gamma_5 \)-Hermitian, but

\[ \gamma_5 D(\mu) \gamma_5 = D^\dagger(-\mu). \tag{96} \]

Consequently, for non-vanishing real \( \mu \) the determinant of the Dirac operator then is complex and one cannot obtain a real Boltzmann weight to perform Monte-Carlo simulations. This is the famous "sign-problem", which is under active investigation [67, 68, 69, 70, 71, 72, 73, 74, 75].
3 Chiral symmetry breaking and its restoration

3.1 Numerical evidence for chiral symmetry breaking

The chiral condensate $\Sigma$ varies under chiral transformations and serves therefore as a convenient order parameter of chiral symmetry breaking, see Sect. 2.5. Chiral symmetry breaking is a non-perturbative phenomenon and can be analyzed on the lattice and in chiral perturbation theory ($\chi$PT). As proposed in Ref. [76] an efficient method to determine the value of $\Sigma$ in lattice calculations is based on the Banks-Casher relation (86), see Ref. [43]. It uses the condensation of low modes of the Dirac operator near the origin. In any numerical determination of a quantity the results of a computation need an extrapolation to certain limits, in this case the limits indicated in Eq. (86). The functional dependence on the extrapolation parameters can be derived within the extended framework of $\chi$PT [77, 78] and the Gell-Mann-Oakes-Renner (GMOR) relation [79]

$$\lim_{m \to 0} \frac{M_{\pi}^2 F_{\pi}^2}{2m} = \Sigma.$$  (97)

According to this relation the derivative of $M_{\pi}^2 F_{\pi}^2/2m$ with respect to the quark mass $m$ in the chiral limit must be equal to this condensation rate. $M_{\pi}$ and $F_{\pi}$ are here the mass and the decay rate of the Nambu-Goldstone bosons. We use the convention $F_{\pi} \approx 90$ MeV and $N_f=2$.

The first step in numerical determination of $\Sigma$ based on the Banks-Casher relation (86) is the calculation (53) of the lowest eigenvalues $\lambda$ of the Dirac operator $D$ for the lightest quark flavor $u$ with mass $m$

$$D u_\lambda = \lambda u_\lambda.$$  (98)

For $D = D_L$ of Eq. (43) the eigenvalues are purely imaginary, for the $\gamma_5$-Hermitian Dirac matrices they appear in complex conjugate pairs or are real, and for fermions obeying the Ginsparg–Wilson relation, $D \gamma_5 + \gamma_5 D = a D \gamma_5 D$, they are distributed on the Ginsparg-Wilson circle. The computer codes usually deliver the absolute values of $\lambda$ in units of the lattice constant $a$. Small values of $\lambda$ are directly related to the central angle $\phi$ of the Ginsparg-Wilson circle in Fig. 1, $\phi_\lambda \approx a \lambda$, see Eq. (83). The density $\rho_\lambda(\phi,m)$ of eigenvalues for given $\phi$ and $m$ is approximately independent of the lattice constant $a$ and increasing with number $V_L$ of lattice sites. The value of the chiral condensate follows from the spectral density near zero via the Banks-Casher relation (86).

Computationally more efficient than the determination of $\rho_\lambda(\phi,m)$ is the integral over the spectral density, the gluonic average over the number of modes in the interval $[-\Lambda, \Lambda]$

$$\nu(\Lambda, m) := \left\langle \sum_{|\lambda|<\Lambda} \right\rangle_G V_L \int_{-a\Lambda}^{a\Lambda} d\phi \rho_\lambda(\phi,m),$$  (99)

It is quite understandable that in the $\Lambda$-region around zero $\nu(\Lambda, m)$ is proportional to $V_L$. For the free case one can derive $\nu(\Lambda, m) \propto V_L \Lambda^4$, from Eq. (99) one is getting $\rho_\lambda(\phi) \propto \phi^3$ and therefore a vanishing condensate (86). In contrast to this result the numerical calculations show that in the chirally broken phase $\nu$ of Eq. (99) grows proportional to $\Lambda$

$$\nu(\Lambda, m) \propto V_L (2a\Lambda) \left( \frac{a^3}{\pi} \Sigma \right) + \cdots = \frac{2V}{\pi} \Lambda \Sigma + \cdots,$$  (100)

where the symmetry of the integration region $[-a\Lambda, a\Lambda]$ explains the factor 2.

Recently, the mode number $\nu(\Lambda, m)$ for two light quark flavors has been computed for the tree-level Symanzik improved gluon action and the Wilson twisted mass fermion action by Refs. [80, 81].
and for the standard Wilson gluonic action and the non-perturbatively O(a)-improved Wilson fermion action [48] by Refs. [82, 83, 7]. As an example for the determination of the chiral condensate on the lattice we follow the discussion in Ref. [7].

The mode number ν(Λ,m) of Eq. (99) is equal to the average number of eigenvalues α of the massive Hermitian Dirac operator \( D^\dagger_m D_m = (D^\dagger_W + m)(D_W + m) \) according Eq. (42) with \( \alpha \leq \Lambda^2 + m^2 \). Expression (99) gives the so called “bare chiral condensate” at a given lattice constant \( a \) only. As QCD is a renormalizable theory the “physical” value of the chiral condensate depends on the regularization scheme and can be converted to other schemes by the appropriate renormalization factors \( Z \). As proven in Ref. [76] the rate of condensation is renormalizable and unambiguously defined after renormalization of the bare action parameters. Therefore, the following results, see also Refs. [84, 76, 82, 83, 7], can be computed using the (improved) Wilson formulation of lattice QCD even though the latter violates chiral symmetry at energies on the order of the inverse lattice spacing.

We conclude that the mode number \( \nu \) is a renormalization group invariant quantity

\[
\nu_R(\Lambda_R,m_R) = \nu(\Lambda,m), \quad \Lambda_R = Z_\mu \Lambda, \quad m_R = Z_m m.
\]  

The renormalized chiral condensate \( \Sigma_R \) can be deduced from the discretized derivative of Eq. (100), by the “effective spectral density”

\[
\tilde{\rho}_R(\Lambda_R,m_R) = \frac{\pi}{2}\nu_R(\Lambda_{R1},m_R) - \nu_R(\Lambda_{R1},m_R), \quad \Lambda_R = \frac{\Lambda_{R2} + \Lambda_{R1}}{2}.
\]

The left diagram in Fig. 2 depicts the result of the determination of the mode number for nine values of \( \Lambda_R \) for given lattice constant \( a \) and two light fermions of mass \( m_R = 12.9 \) MeV \( (m_R/m^\Lambda_{\overline{\text{MS}}} \approx 0.126 \) according to Eq. (7.9) of Ref. [85] on a \( 64^3 \times 128 \) lattice. The renormalized values are given in the \( \overline{\text{MS}} \)-scheme at the renormalization scale \( \mu = 2 \) GeV. As the quadratic fit to the data \( \nu_R = -9.0(13) + 2.07(7)\Lambda_R/\text{MeV} + 0.0022(4)(\Lambda_R/\text{MeV})^2 \) shows, the main contribution to \( \nu_R \) is linear in \( \Lambda_R \) while the constant and quadratic term are in the investigated region of the order of 10%. This agrees with the expectation from the Banks–Casher relation.

![Figure 2](image-url)

Figure 2: Left: The mode number \( \nu_R(\Lambda_R,m_R) \) as a function of \( \Lambda_R \) with a quadratic fit to the data \( \nu_R = -9.0(13) + 2.07(7)\Lambda_R/\text{MeV} + 0.0022(4)(\Lambda_R/\text{MeV})^2 \). \( m_R/m_c \approx 0.14 \). Right: Linear fits in \( a^2 \) to the effective spectral density \( \tilde{\rho}_R(\Lambda_R,m_R) \) for three values of \( \Lambda_R \) and fixed \( m_R \). Courtesy of [83].

From the eight couples of consecutive values of \( \Lambda_R \) eight values of the effective spectral density \( \tilde{\rho}_R \) are determined with Eq. (102) for the three considered values of \( m_R \) and three lattice spacings \( a \) in Ref. [83]. This allows at fixed quark masses \( m_R \) and averaged \( \Lambda_R \)-values a fit with a linear
The chiral limit, on the other hand, requires an assumption on how \( \Sigma \) behaves for \( m_R \to 0 \). A corresponding framework is given by chiral perturbation theory (\( \chi PT \)). \( \chi PT \) is an effective theory based on the spontaneous breaking of chiral symmetry and a soft explicit breaking by quark-mass terms. \( \chi PT \) predicts in next to leading order \( \rho_R \to \Lambda_R \)-independent [83]. By different fits inspired from \( \chi PT \) Ref. [83] extrapolates \( \rho_R(\Lambda_R, m_R) \) to the chiral and the continuum limit, see Fig. 4, and finds for the chiral condensate values of \( \Sigma^{1/3} = 261(6) \) MeV, 253(9) MeV and by fitting the data in all three directions \( \Lambda_R, m_R \) and \( a \) at the same time 259(6) MeV. These results are in good agreement with earlier investigations in quenched lattice QCD with exact chiral symmetry [86], where a chiral condensate of \( \Sigma^{1/3} = 250(3) \) MeV in the \( \overline{\text{MS}} \)-scheme at the renormalization scale 2 GeV was determined.

After this detailed discussion of a lattice determination of the chiral condensate \( \Sigma \) we would like to present as a result of the recent compilation [87] the estimates of the \( N_f = 2 \) and \( N_f = 2 + 1 \) condensates in the \( \overline{\text{MS}} \)-scheme at the renormalization scale 2 GeV

\[
N_f = 2 : \quad \Sigma^{1/3} = 266(10) \text{ MeV} \quad \text{Refs. [83, 81, 88],}
\]
\[
N_f = 2 + 1 : \quad \Sigma^{1/3} = 274(3) \text{ MeV} \quad \text{Refs. [89, 90, 91, 92].}
\]

The errors include both statistical and systematic uncertainties.
Figure 4: Effective spectral density $\rho_R$ versus the cutoff $\Lambda_R$ in the continuum and chiral limits. The constant line gives the value for the chiral condensate. Courtesy of [83].

More recently, the authors of [93] compute the chiral condensate in 2+1-flavor QCD through the spectrum of low-lying eigenmodes of Dirac operator. The number of eigenvalues of the Dirac operator is evaluated using a stochastic method with an eigenvalue filtering technique on the background gauge configurations generated by lattice QCD simulations including the effects of dynamical up, down and strange quarks described by the Möbius domain-wall fermion formulation. The spectrum shape and its dependence on the sea quark masses calculated in numerical simulations are consistent with the expectation from one-loop chiral perturbation theory. After taking the chiral and continuum limits using the data at three lattice spacings ranging $0.080 - 0.045$ fm, they obtain $\Sigma^{1/3} = 270(4.9)$ MeV, with the error combining statistical and various sources of systematic errors. Finite volume effects are confirmed to be under control by a direct comparison of the results from two different volumes at the lightest available sea quarks corresponding to 230 MeV pions. JLQCD and TWQCD Collaborations [94, 95] find slightly lower values for the chiral condensate using lattice QCD, chiral Random Matrix Theory and chiral perturbation theory with $N_f = 2, 2+1$ and 3.

In the last few years, the Yang-Mills gradient flow was shown to be an attractive tool for non-perturbative studies of non-Abelian gauge theories. In view of its renormalization properties [96, 97], and since its application in lattice gauge theory is technically straightforward, the gradient or Wilson flow allows the dynamics of non-Abelian gauge theories to be probed in many interesting ways. The flow can be used for accurate scale setting, for example, and it provides an understanding of how exactly the topological (instanton) sectors emerge in the continuum limit of lattice QCD [96]. Moreover, observables at positive flow time are natural quantities to consider for non-perturbative renormalization and step scaling [98, 99, 100, 101]. Matter fields may or may not be included in the flow equations. A fairly trivial extension of the flow to the quark fields in QCD is achieved, by leaving the flow equation for the gauge field unchanged, while the evolution of the quark fields as a function of the flow time is determined by a gauge-covariant heat equation. Ref. [102] gives an excellent introduction and overview of the gradient flow in QCD and illustrates two applications of the extended flow with respect to chiral symmetry, one being a new strategy for the calculation of the axial-current renormalization constant in lattice QCD and the other a computation of the chiral condensate essentially through the evaluation of the expectation value of the scalar quark density at positive flow time. In both cases, the method is technically attractive, the chiral condensate, for example, is easily obtained with high precision, because no additive renormalization is required.
3.2 Restoration of Chiral Symmetry

In order to understand the mechanisms of chiral symmetry breaking, it is important to study how chiral symmetry can be restored. On general grounds [103] it is expected that at zero light quark mass there is a second-order phase transition in QCD for which the chiral condensate is the order parameter. This is true under an additional assumption that the $U(1)_A$ axial symmetry breaking is still large at the chiral critical temperature, $T^*$, and the symmetry is restored at higher temperatures. It has been established by lattice QCD calculations that at the physical values of light quark masses and at vanishing chemical potential there is no genuine phase transition in QCD, but rather a “rapid” crossover [104, 105, 106]. While the low-temperature phase exhibits confinement and breaking of chiral symmetry, at high temperatures the behavior of the theory is qualitatively different – the interaction between quarks and gluons decreases due to asymptotic freedom, leading to deconfinement, and the chiral symmetry is restored (see [107] for a recent review). In the crossover region the QCD partition function does not exhibit a singularity, so it is not a surprise that different physical observables show a change in their temperature dependence at somewhat different temperatures. Chiral symmetry restoration has been clearly observed in high precision lattice calculations at high temperature and vanishing chemical potentials, as an analytic crossover close to the deconfinement crossover, see e.g., [108, 104, 109, 105, 106, 110, 111, 112]. While studies based on the Wilson [113, 114] or chiral fermion formulations [115] are, at present, constrained to a regime of moderately light quark masses ($m_l/m_s>0.2$), with recent improvements in [116], calculations exploiting staggered fermion discretization schemes [104, 109, 105, 117, 118, 119, 120, 121, 122, 123] can be performed with an almost realistic spectrum of dynamical light and strange quarks.

If the physical light quark mass is small enough one may hope that remnants of criticality (plus subleading corrections) still govern the crossover region. Applicability of the critical scaling was extensively studied in [124] with staggered fermions and the scaling was indeed observed. The HotQCD collaboration calculated the light quark condensate and its susceptibility with the asqtad and HISQ/tree actions on lattices with the temporal extent $N_T = 6, 8$ and 12 at several values of the light quark mass and then performed fits to $O(N)$ scaling functions complemented by non-singular terms. The chiral condensate $\langle \bar{\psi}\psi \rangle$ requires both multiplicative and additive renormalizations at finite quark masses, hence results from the subtracted chiral condensate $\Delta_{ls}(T)$ and according disconnected susceptibility as introduced in Ref. [118] are plotted in Fig. 5 for various improved staggered fermion representations. The pseudo-critical temperature, $T_{pc}$, defined this way as a location of the peak of the chiral susceptibility, reduces to the true critical temperature in the chiral limit. The result in the continuum limit at the physical light quark mass is $T_{pc} = 154(9)$ MeV [111]. This is compatible with earlier results by the Budapest-Wuppertal collaboration that are in a range $T_{pc} = 147 - 157$ MeV, depending on what chiral observable is picked to determine $T_{pc}$ [123].

It was noted by Pisarski and Wilczek that the order of the chiral phase transition in QCD may depend on the number of light quark degrees of freedom and qualitative features of the transition may also change with the quark mass [103]. In QCD with 3 massless quark flavors the chiral phase transition is expected to be first order. If this is the case, the phase transition remains first order even for non-zero values of the quark masses and terminates at a critical quark mass $m_q^{cr}$, or equivalently at a critical pion mass, $m_{\pi}^{cr}$, where the transition becomes second order belonging to the 3-d $Z(2)$ Ising universality class. For quark mass $m_q > m_q^{cr}$ chiral restoration takes place through a smooth crossover. It is quite important with respect to the discussion of $m \to \infty$, when the Polyakov line becomes an order parameter, and $m \to 0$, when $\langle \psi \bar{\psi} \rangle$ becomes an exact order parameter, to investigate the quark mass dependence of the phase diagram and the nature of the transition. For recent results and detailed discussion consult [125, 126, 127] and references therein. For zero chemical potential, the values of critical quark masses characterize a line of chiral phase transitions in the light-strange quark mass plane. This line extends toward the non-zero chemical potential...
Figure 5: The subtracted chiral condensate and disconnected susceptibility [118] for the asqtad and HISQ/tree staggered actions with $m_l = m_s/20$ and after an interpolation to the physical light quark mass using the $m_l/m_s = 0.05$ and $0.025$ data (black diamonds) is compared with the continuum extrapolated stout action results [123]. The temperature $T$ is converted into physical units using the kaon decay constant $f_K$ to set the lattice scale. Courtesy of [111].

direction and forms a surface of phase transitions in the 3-d, Z(2) universality class. The chemical potential where this surface intersects the physical values of light and strange quark masses may correspond to the QCD critical point [128]. However, determining the curvature of this surface turns out to be complicated [129, 130] and, in fact, is likely to suffer from similar lattice cut-off effects as those contributing to the value of the critical pion mass itself. Other possibilities for generating a second order transition at the physical values of quark masses have been discussed in [131]. The first order chiral phase transition in 3–flavor QCD has been investigated on coarse lattices using unimproved [132, 133, 134, 135] as well as improved actions [136, 128, 137, 104, 138, 139, 140]. However, on these coarse lattices the critical pion mass value turns out to be strongly cut-off and regularization scheme dependent. As of now no continuum extrapolated results exist. Current results for the critical pion mass obtained in calculations with staggered (standard and p4fat3) fermions on $N_t = 4$ and 6 lattices vary from about 300 MeV down to about 70 MeV [132, 133, 134, 135, 136, 128, 137, 104, 138]. While studies using the clover improved Wilson fermion action on $N_t = 4$, 6, 8 and 10 lattices suggest that $m_c^\pi$ can change from about 750 MeV to about 100 MeV [139, 140]. In general it is found that the critical pion mass decreases when using either improved actions or when reducing the lattice spacing. A study of 4-flavor QCD using HYP action [141] also suggests that the first order chiral phase transition becomes weaker in the continuum limit.

Concerning large density, new chiral phases are expected [142, 143, 144, 145], but they are hard to observe because of the sign problem. In the presence of a finite chemical potential the fermion determinant becomes complex and standard Monte-Carlo techniques are not applicable. In recent years, progress was made by a combination of numerical simulations, mostly for small chemical potential, analytical techniques and investigation of model systems, see [146, 74] for some review. In [147] it was shown that $SU(N_c)$ lattice gauge theory with $N_f$ flavors of massless staggered fermions in the strong coupling limit, i.e., at vanishing inverse bare gauge coupling $\beta$, is in a chirally symmetric phase provided the quark chemical potential $\mu$ is large enough. This is an exact result obtained by means of a convergent cluster expansion and it was extended into the region of non-vanishing $\beta$ in [148]. The strong coupling limit in the presence of chemical potential has been investigated in the literature, mostly for the cases of $SU(2)$ and $SU(3)$ and $N_f = 1$, in a variety of approaches.
One approach relies on integrating out the gauge field, resulting in a representation of the partition function in terms of monomers, dimers and baryon loops [149], or monomers, dimers and polymers [150, 151]. The sign problem is partly evaded within this representation, thus allowing simulations. In such simulations in the case of \( N_c = 3 \) [150] a chiral symmetry restoring first order transition was found at some critical \( \mu \). Similarly, for \( N_c = 2 \), restoration of chiral symmetry at large \( \mu \) and/or \( T \) was observed in [151].

More recently, the two-color \((N_c = 2), N_f = 1\) case was investigated in [152] using the dimer-baryon loop representation with a new updating algorithm [153]. A second order transition to a chirally symmetric phase at some critical \( \mu \) was seen in good agreement with mean field predictions. In the case of \( N_c = 3 \) such improved simulations were carried out in [154]. Another approach is based on mean field investigations of effective actions. In [155] [156, 157] an effective action was obtained by performing a \( 1/d \) expansion in the spatial directions and retaining only the leading terms, while leaving the time-like directions intact. This effective action was used to obtain the phase diagram in \( T, \mu \) and quark mass \( m \) within a mean field approximation which exhibits a chiral phase at large \( \mu \) and/or \( T \). Further results on the chirally restored phase can be found in [158, 159, 160, 161, 162, 163] and the the chiral restoration with chemical potential was also studied in [164, 165, 166, 167, 168, 169, 170, 171, 172]. Finite volume effects were investigated in [173, 174, 175] via finite size scaling. Moreover, chiral symmetry is the remaining symmetry in the conformal window of technicolor models, discussed in [176], investigating particle spectra.

An interesting toy model without the sign problem is QCD with a magnetic field. ‘Magnetic catalysis’ refers to an increase of the condensate with \( B \) implying a \( B \)-dependence of \( T_c \) as well. Almost all low-energy models and approximations to QCD as well as lattice simulations in quenched theories [177, 178] and at larger than physical pion masses in \( N_f = 2 \) QCD [179, 180, 181, 182] and in the \( N_f = 4 \) SU(2) theory [183] found \( \bar{\psi}\psi(B) \) and \( T_c(B) \) to increase with \( B \). In contrast to the above results, a large-scale study of QCD in external magnetic fields with physical pion mass \( M_\pi = 135 \text{ MeV} \) [184] and results extrapolated to the continuum limit [185] has revealed the transition temperature to decrease as a function of the external magnetic field. This applies to the \( T_c \)’s defined from the quark condensate, the strange quark number susceptibility and the chiral susceptibility. In particular, they found the condensate to depend on \( B \) in a non-monotonous way in the crossover region. Two rationales why former lattice studies are at variance with these recent findings: coarser lattices and larger quark masses, are pointed out in [185]. Obviously, it is also very important to address the differences between our QCD results and many model and chiral perturbation theory predictions, especially since the latter methods can be used to investigate regions that are not easily accessible to lattice simulations, e.g., QCD at a non-vanishing baryon density. For more details and results on the so-called ‘chiral magnetic effect’ (CME) see Sect. 4.8.

Finally, chiral restoration can also be investigated with high precision via the modifications of the hadron spectrum in nuclear matter [186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202]. Chiral symmetry is broken for the hadronic groundstates, but its restoration is expected for highly excited hadrons, revealing some parity doubling in the \( u, d \) sector which is conjectured to reflect effective restoration of chiral symmetry in the vacuum. The conjecture depends on a generalization of the Banks-Casher theorem, exploiting the fact that all dynamics sensitive to observables of spontaneous chiral symmetry breaking in correlation functions arise from fermion modes of zero virtuality (in the infinite volume limit), while such modes make no contribution to any of the dynamics which preserve chiral symmetry. Whether this conjecture is correct or not can also be answered experimentally since the conjectured symmetry requires existence of some not yet observed states. Recent and most complete experimental analysis on highly excited nucleons that includes not only elastic \( \pi N \), but also the photo-production data, does report evidence for some of the missing states and the parity doubling patterns look now even better than before [203, 204, 205, 206, 207, 208]. See also [209] for an excellent review on pion physics.
4 Mechanism of chiral symmetry breaking

Quantum chromodynamics (QCD) at low energies is dominated by the non-perturbative phenomena of quark confinement and (spontaneous) chiral symmetry breaking (\(\chi\)SB). A rigorous treatment of them is only possible in the lattice regularization and the interplay between \(\chi\)SB and confinement as well as the chiral and deconfinement phase transitions at finite temperature and density are of continuous interests [210, 211, 212, 43, 213, 214, 215, 216, 217].

The origin of \(\chi\)SB may be described as an analog to magnetization. Its strength is measured by the fermion (chiral) condensate in Eq. (78), which is an order parameter for \(\chi\)SB. It is a vacuum condensate of bilinear expressions involving the quarks in the QCD vacuum. The Banks-Casher relation (86) links the spectral density of the Dirac operator with the existence of a chiral condensate and spontaneous \(\chi\)SB [43], see also Sect. 2.5. In particular, it relates the chiral condensate to the density \(\rho(0)\) of near-zero Dirac eigenmodes, i.e., low-lying nonzero eigenmodes \(|\lambda\rangle\) of the Dirac equation \(D|\lambda\rangle = \lambda |\lambda\rangle\), distributed around \(\lambda = 0\). Hence, \(\chi\)SB should be imprinted in the chiral properties of the near-zero modes.

We start with reviewing the latest investigations on the interplay between confinement and \(\chi\)SB by various groups and look at the spectral density and chiral properties of the low-lying eigenmodes. Next, we investigate the origin of these near-zero modes from topological models, looking at instantons and center vortices in particular. Furthermore, we analyze the dimensionality of the low-lying eigenmodes and take a look at the fractality of center domains. Finally we discuss the chiral magnetic effect and the Green's Function approach to \(\chi\)SB and relations to lattice QCD and topological configurations.

4.1 A Link between chiral symmetry breaking and confinement

According to Casher’s argument, a force strong enough to confine quarks is also generally expected to break chiral symmetry [212]. At the QCD finite temperature transition chiral symmetry is restored and the theory deconfines. There are some indications from numerical simulations in lattice QCD that the (pseudo-)critical temperature \(T_c\) is the same for both transitions. Thus it is widely believed that there must be a mechanism linking the two phenomena and it is an interesting check whether field configurations with restoration of chiral symmetry still have confinement and vice versa. Concerning confinement no signature in spectral properties of the Dirac operator was known, on the other hand it is obvious that such signatures must exist, since the quark propagator, clearly knows about confinement properties.

The QCD transitions are characterized by the breaking and restoration of center and chiral symmetry, which are well defined in two extreme quark mass limits, respectively. In the chiral limit, where the current quark mass is zero \(m = 0\), the chiral condensate is the order parameter for the chiral phase transition. When the current quark mass goes to infinity \(m \rightarrow \infty\), QCD becomes pure gauge \(SU(3)\) theory, which is center symmetric in the vacuum and the order parameter that is generally used is the Polyakov loop [210],

\[
P(\vec{x}) = \frac{1}{N} \text{Tr} \prod_{x_4=1}^{N_T} U_4(\vec{x}, x_4),
\]

(104)

the product of temporal link variables \(U_4(x)\) along closed loops that wind around the compact time direction. A Polyakov loop can be thought of as the world-line of a massive static quark at spatial position \(\vec{x}\), propagating only in the periodic time direction. Hence, its expectation value

\[
\langle P(\vec{x}) \rangle = e^{-aF_4N_t}
\]

(105)
can be related to the free energy of an isolated quark $F_q$, which in the confinement phase is infinite, while it is finite in a non-confined phase. Therefore, the Polyakov loop is a true order parameter: zero in one phase, non-zero in another, which associates the breaking of a global symmetry with the transition from one phase to another, i.e., the global center symmetry of the gauge field [218].

In order to link confinement to spectral properties of the Dirac operator, the authors of [219, 220, 221, 222, 223, 224, 225, 226] represent Polyakov loops and their correlators as spectral sums of eigenvalues and eigenmodes of the lattice Dirac operator. The deconfinement transition of pure gauge theory is characterized by a change in the response of moments of eigenvalues of the Dirac operator to varying the boundary conditions. They argue that the potential between static quarks is linked to spatial correlations of Dirac eigenvectors, by introducing a new QCD phase transition order parameter, the dual condensate, which connects confinement and $\chi$SB as different mass limits.

Starting with the chiral condensate in Eq. (78) they introduce the generalized temporal boundary conditions $\psi(\vec{x}, \beta) = e^{i\phi} \psi(\vec{x}, 0)$ for the fermion fields $\psi$, where the canonical choice is anti-periodic, i.e., $\phi = \pi$, while they allow arbitrary values $\phi \in [0, 2\pi]$, indicated by a subscript for the Dirac operator. They define the gauge invariant "dual quark condensate" $\Sigma_n$ as the Fourier transform with respect to $\phi$,

$$\tilde{\Sigma}_n(m, V) = \int_0^{2\pi} d\phi \frac{e^{-i\phi n}}{2\pi} \left\langle \text{Tr} \left[ (m + D\phi)^{-1} \right] \right\rangle_G,$$

where the index $n$ is an integer, which can be related to equivalence classes of Polyakov loops. Therefore, the propagator (106) is expressed as a geometric series

$$\text{Tr} \left[ (m + D\phi)^{-1} \right] = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{m^k} \text{Tr} \left[ (D\phi)^k \right].$$

Considering Dirac operators containing only terms that connect nearest neighbors for simplicity, the power $(D\phi)^k$ corresponds to a chain of $k$ hops, picking up a gauge link every single time. The trace in Eq. (107) is over color- and space-time indices, where the latter implies that the chains of hops have to form closed loops $l$. Therefore, the sum in Eq. (107) can be re-written as a sum over the set $\mathcal{L}$ of all possible closed loops on the lattice. Considering a lattice with even numbers of sites in all directions, we have

$$\text{Tr} \left[ (m + D\phi)^{-1} \right] = \frac{1}{m} \sum_{l \in \mathcal{L}} e^{i\phi q(l)} \text{Tr}_c \prod_{(x, \mu) \in l} U_\mu(x).$$

The remaining trace $\text{Tr}_c$ is over the color indices of the ordered product of all link variables $U_\mu(x)$ along a loop $l$, of length $|l|$. Loops wrapping around the temporal boundary $q(l)$ times, where the winding number $q(l) \in \mathbb{Z}$, pick up a factor of $\exp(\pm i\phi q(l))$ if they run forward or backward in time, respectively. When the expression (108) is inserted into the formula (106) for the dual condensate, the $\phi$-integration with the additional Fourier factor $\exp(-i\phi n)$ projects to loops of a particular winding number $n$ and the sum now runs over the set $\mathcal{L}(n)$ of loops that wind $n$-times around the compact time direction

$$\tilde{\Sigma}_n(m, V) = \frac{1}{Vm} \sum_{l \in \mathcal{L}(n)} \frac{s(l)}{(2am)^{|l|}} \left\langle \text{Tr}_c \prod_{(x, \mu) \in l} U_\mu(x) \right\rangle_G.$$

The case of $n = 1$, i.e., the dual condensate $\tilde{\Sigma}_1(m, V)$ which corresponds to loops that wind exactly once, is referred to as the "dressed Polyakov loop". It is obvious from Eq. (109) that in the large-$m$ limit the dominant contribution is the conventional thin Polyakov loop (as this is the shortest loop winding once). Under a center transformation, $U_4(\vec{x}, t_0) \to zU_4(\vec{x}, t_0)$, where all temporal links on a time-slice, i.e., at some fixed $t_0$, are multiplied with an element $z$ of the center of the gauge group, the dressed Polyakov loop, i.e., the dual condensate for $n = 1$, transforms as $\tilde{\Sigma}_1 \to z\tilde{\Sigma}_1$, which is the same
transformation law as for the thin Polyakov loop. Hence, an order parameter for center symmetry and/or confinement is obtained, which at the same time, after performing the consecutive limits of infinite volume and vanishing mass, \( \Sigma = \lim_{m \to 0} \lim_{V \to \infty} \Sigma(m,V) \), acts as a proper order parameter for \( \chi_{\text{SB}} \).

The dual condensate \( \Sigma \) can also be expressed as a spectral sum over all Dirac eigenvalues \( \lambda_{\phi}^{(i)} \) (see also Eq. (53), again evaluated for the boundary angle \( \phi \)),

\[
\tilde{\Sigma}_n(m,V) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i\phi n} \sum_{i} \left( \langle m + \lambda_{\phi}^{(i)} \rangle^{-1} \right) G.
\] (110)

A Banks-Casher type of representation relates the chiral condensate to the density of eigenvalues at the origin also for an arbitrary boundary angle \( \phi \). The dual condensate is then obtained by integrating the \( \phi \)-dependent spectral density \( \rho(\phi) \)

\[
\tilde{\Sigma}_1 = \int_0^{2\pi} \frac{d\phi}{2} e^{-i\phi} \rho(\phi),
\] (111)

for the case of \( n = 1 \). Below \( T_c \) the spectral density at the origin is constant as a function of \( \phi \) and a vanishing dressed Polyakov loop emerges. More interesting is the situation above \( T_c \), where a non-trivial \( \phi \) dependence is necessary for a non-vanishing dressed Polyakov loop. Naively one would think that above \( T_c \) the spectral density at the origin must be zero, such that the chiral condensate may vanish. However, in [227] (for a different phase convention) it was shown that the spectral gap depends on the relative phase between the boundary angle \( \phi \) and the phase \( \theta \) of the Polyakov loop.

![Figure 6](image_url)

Figure 6: (left) The integrand \( I(\phi) = V^{-1} \sum_{i} \langle m + \lambda_{\phi}^{(i)} \rangle^{-1} \) of (110) in lattice units for two values of am. The data are from 20 gauge configurations on \( 12^3 \times 6 \) lattices below \( T = 255 \text{ MeV}, a = 0.129 \text{ fm} \) and above \( T_c \) (\( T = 337 \text{ MeV}, a = 0.098 \text{ fm} \)). (right) The dressed Polyakov loop at \( m = 100 \text{ MeV} \) in units of GeV\(^3\) as a function of the temperature \( T \) in MeV. Courtesy of [222].

From Fig. 6 (left) it is obvious that below \( T_c \) the integrand \( I(\phi) = V^{-1} \sum_{i} \langle m + \lambda_{\phi}^{(i)} \rangle^{-1} \) of (110) is essentially constant, while above \( T_c \) it shows a pronounced cosine type of behavior. Integrating over \( \phi \) with the weight \( \exp(-i\phi) \), i.e., \( n = 1 \), gives a vanishing dressed Polyakov below \( T_c \), while above \( T_c \) a non-vanishing value is observed, see Fig. 3 (right), hence the transition from confinement to deconfinement leads to a different response of the spectral sums to the changing temporal fermion boundary conditions. It was also shown in [228], that the IR modes play the dominant role in this.
Further numerical results of the relation between the fermion spectrum at general boundary conditions, the dual condensate and dressed Polyakov loops for (un-)quenched and dynamical $SU(3)$ lattice configurations were presented in e.g., [220, 221, 229, 230, 231], confirming a direct link between confinement and $\chi_{SB}$.

Furthermore, by using the dressed Polyakov loop or dual chiral condensate as an equivalent order parameter, Huang et. al. [232] investigated the relation between the chiral and deconfinement phase transitions at finite temperature and density in the framework of three-flavor Nambu–Jona-Lasinio (NJL) model. It is found that the behavior of dressed Polyakov loop in absence of any confinement mechanism still shows an order parameter like behavior. In the chiral limit, the critical temperature for chiral phase transition coincides with that of the dressed Polyakov loop in the whole $(T,\mu)$ plane, but in the case of explicit $\chi_{SB}$ the phase transitions are flavor dependent. The transition temperature of the dressed Polyakov loop $T_{c}^{\varphi}$ is larger than that of chiral restoration $T_{c}^{\chi}$ for all flavors in the low baryon density region where the transition is a crossover, whereas, in the high baryon density region the phase transition is of first order and the two critical temperatures coincide. Therefore, there are two critical end points, i.e., $T_{cep}^{ud}$ and $T_{cep}^{su}$ at finite density. They also explain the feature of $T_{c}^{\chi} = T_{c}^{\varphi}$ in the case of 1st and 2nd order phase transitions, and $T_{c}^{\chi} < T_{c}^{\varphi}$ in the case of crossover. It is expected that this feature is general and can be extended to full QCD [232].

### 4.2 Dirac mode expansion of the Polyakov loop

Expressing gluonic observables in terms of spectral sums and truncating these sums is sometimes referred to as “low eigenmode filtering”. It makes use of the fact that the low-lying modes of the Dirac operator are an efficient filter for infrared properties of the gauge field. In a thermalized configuration, i.e., one not treated with cooling or smearing, these are hidden under UV fluctuations and a filter is needed to observe them. Using the low-lying Dirac modes has the advantage over smearing or cooling techniques, that the gauge field is not altered. The breaking of locality and reflection positivity by such a filtering construction should not be an issue in the analysis of gluonic variables, but it might be problematic regarding the interpretation of results of the quark propagator and hadron spectra in the next section.

Using the arguments from the last section (in more detail in [219]), the Polyakov loop can be expressed as a spectral sum. The hopping terms of the Dirac operator $D$, e.g., Eq. (42), connect nearest neighbors. When powers of $D$ are considered, these terms combine to chains of hops on the lattice and products of the link variables $U_\mu(x)$ are collected along these chains, see Eq. (108). In particular, considering the $N_t$-th power of $D$, where $N_t$ is the temporal extent of our lattice one gets chains with a maximum length of $l = N_t$. Setting the two space-time arguments of $D$ to the same value, $y = x$, only closed chains, i.e., loops starting and ending at $x$ are considered and among these are the loops where only hops in time direction occur such that they close around compact time, i.e., Polyakov loops. Using the fact that the Polyakov loops respond differently to a change of the boundary conditions compared to other, non-winding loops, it is easy to see that combining periodic and anti-periodic boundary conditions (subscripts $\pm$) one can project out the Polyakov loops, expressed as a spectral sum over eigenvalues $\lambda$,

$$P = \frac{1}{8V_L}\left(2\sum_{\lambda} \lambda^N - (1+i)\sum_{\lambda^+} \lambda^N_{\lambda^+} - (1-i)\sum_{\lambda^-} \lambda^N_{\lambda^-}\right).$$  

(112)

$V_L = N_s^3N_t$ is the total number of lattice points and the usual average over spatial sites to improve statistics is already considered. The equation expresses the Polyakov loop through moments of the Dirac eigenvalues at different boundary conditions and the individual sums are real since the eigenvalues come in complex conjugate pairs.
The interesting question is which part of the Dirac spectrum carries most of the signal for the Polyakov loop. The infrared part of the spectrum (small eigenvalues) is known to undergo a pronounced change as one crosses from the confining to the deconfined phase: A gap opens up in the spectrum, the density of eigenvalues near the origin vanishes and, according to the Banks-Casher formula, chiral symmetry is restored. Although the low-lying eigenvalues undergo a dramatic change, it is not clear whether they give a sizable contribution to the Polyakov loop: Due to the large power $N_t$, the small eigenvalues are strongly suppressed relative to the bulk of the spectrum where eigenvalues of $O(1)$ occur (in lattice units). It was shown numerically in [219], that summing up the lowest 100 eigenvalues of the chirally improved lattice Dirac operator [65, 66] for quenched $SU(3)$ configurations on $20^3 \times 6$ lattices only gives a small fraction of the true Polyakov loop $P$ as determined directly from the link variables for temperatures below and above $T_c$. It is remarkable, however, for $T > T_c$, the spectral sum (112) evaluated with even only the 50 lowest eigenvalues already gets the phase of the Polyakov loop right (the $Z_3$ symmetry gives rise to 3 preferred phases of the Polyakov loop expectation value). In [220] it is shown for the staggered Dirac operator [50, 51], that the Polyakov loop gets its main contributions from the UV end of the spectrum.

Suganuma et al. [233, 234, 235, 236, 237, 238] use the same approach and cut the eigenvalue spectrum from below. They investigate the Polyakov loop and its fluctuations with eigenmodes of the staggered Dirac operator and find that the low-lying Dirac modes yield negligible contributions to the Polyakov loop fluctuations. This property is confirmed to be valid in confined and deconfined phase by numerical simulations in $SU(3)$ quenched QCD. To differentiate the importance of the low-lying Dirac modes on the properties of the Polyakov loop fluctuations and the chiral condensate, they introduce an infrared cutoff $\Lambda$ and sum only over eigenvalues $|\lambda_n| \geq \Lambda$ and form ratios of the cutoff-dependent chiral condensate $R_{\text{chiral}}$ and Polyakov loop fluctuations $R_{\text{conf}}$ with respect to their full values. The chiral condensate is strongly affected by the low-lying Dirac modes. Taking a typical value for the infrared cutoff $\Lambda \simeq 0.4 \text{ GeV}$ and the quark mass $m \simeq 5 \text{ MeV}$, leads to a drastic reduction of the chiral condensate $R_{\text{chiral}} = \langle \bar{u}u \rangle_{\Lambda} / \langle \bar{u}u \rangle \simeq 0.02$, in a confined phase at $T \simeq 0$ [235, 236, 237]. At the same time the ratio $R_{\text{conf}} \simeq 1$ stays the same, hence the low-lying Dirac modes below the cutoff $\Lambda$ have a negligible contribution to the Polyakov loop fluctuations. In contrast to $R_{\text{chiral}}$, the $R_{\text{conf}}$ ratio is almost unchanged when removing the low-lying Dirac modes even with relatively large cutoff $\Lambda \simeq 0.5 \text{ GeV}$. In earlier works [239, 240, 241], the removal of low-lying eigenmodes conserved the area law behavior of Wilson loops without modifying the slope. They conclude that these results indicate no direct, one-to-one correspondence between confinement and in QCD in the context of different properties of the Polyakov loop fluctuation ratios. Introducing an artificial eigenvalue gap around zero, restores chiral symmetry via Banks-Casher by hand while leaving confinement properties unaltered. This seems to indicate some independence of confinement from chiral properties in QCD.

In [229] the relation of the Polyakov loop to spectral sums of the Dirac-Wilson operator are further investigated by generalizing the approach by Gattringer [219] to mode sums which reconstruct the Polyakov loop locally. This allows to study the Polyakov loop correlator and the static quark anti-quark potential in the light of a few low lying modes of the Dirac operator. The approach is not only valid for lattice Dirac operators but also for the continuum formulation of Yang-Mills theories, and the convergence of the mode sum is demonstrated explicitly. The mode sums are then calculated in the Schwinger model and $SU(2)$ gauge theory, showing that the mode sums are proportional to the Polyakov loop for each point in space. Further, the IR dominated mode sums are considered and the mode sum approximation to the static quark anti-quark potential is obtained numerically. Good agreement between the mode sum approximation and the static potential at large distances for the confinement and the high temperature plasma phase was found.
4.3 Hadron spectra and low-lying Dirac modes

While in the above “low eigenmode filtering” approach gluonic observables were inspected, now the quark propagator is under investigation. After suggestions within the instanton liquid model [242] the effect of the low-lying chiral modes on the $\rho$ and other correlators was studied on the lattice. In a series of papers [243, 244, 245, 246] it was shown that low modes saturate the pseudoscalar and axial vector correlators at large distances and do not affect the part where high-lying states appear. In [246, 247] low mode saturation and also effects of low mode removal for mesons were studied for quenched configurations with the overlap Dirac operator [35, 37]. Subsequently low modes were utilized to improve the convergence of the determination of hadron propagators [246, 247, 248, 249, 250, 251] studying the efficiency when using the low modes of the Dirac operator or the Hermitian Dirac operator.

It is an interesting question whether hadrons and confinement will survive after artificially removing the quark condensate from the vacuum. This can be achieved via removal of the low-lying eigenmodes of the Dirac operator, which is a well defined procedure [246, 197] referred to as “unbreaking” chiral symmetry, that has been addressed in a series of papers [252, 253, 254, 255, 256, 257, 258]. The removal of the low modes (or the chiral condensate) from the quark Green’s function is expected to result in the loss of dynamically generated mass in the valence quarks. Thus, the naive expectation suggests rather light hadrons once the chiral symmetry has been restored. The Graz group studied hadron spectra after cutting low-lying Dirac modes from the valence quark sector in a dynamical lattice QCD calculation. They expressed the valence quark propagators $S$ directly by the eigenfunctions of the Dirac operator and removed an increasing number $k$ of lowest Dirac modes $|\lambda\rangle$

$$S_{\text{red}(k)} = S - \sum_{\lambda \leq k} \mu_\lambda^{-1} |\lambda\rangle \langle \lambda | \gamma_5,$$

(113)

with $\mu_\lambda$ the (real) eigenvalues of the Hermitian Dirac operator $D_5 = \gamma_5 D$. Except for the pion, the hadrons survived this artificial restoration of chiral symmetry by this truncation. The quality of the exponential decay of the correlators increases by this procedure indicating a state with the given quantum numbers. In Fig. 7 the influence of the truncation of the masses of two mesons which can be transformed into each other by a chiral rotation, the vector meson $\rho$ and the axial vector meson $a_1$. These would-be chiral partners become degenerate after restoration of chiral symmetry. Interestingly these meson masses increase with increasing truncation level $k$.

![Figure 7: Influence of the removal of the lowest $k$ modes of the Dirac operator on the masses of chiral partner mesons, the vector meson $\rho$ and axial vector meson $a_1$. Courtesy of [252].]
These results demonstrate that even without a chiral symmetry breaking ($\chi_{SB}$) vacuum confined hadrons can exist, at least with rather large mass. Further, they extract the quark wave-function renormalization function $Z_L(p)$ and the quark mass function $M_L(p)$ from the reduced propagators of Eq. (113) while varying the Dirac low-mode reduction level $k$ from 2 to 512. The dressing functions from the full propagators together with the dressing functions from the most extreme reduction (512 eigenmodes subtracted) are presented in Fig. 8. The plot reveals amplification of infrared suppression of $Z_L(p)$ when subtracting Dirac low-modes while the range from medium to high momenta becomes not affected at all. Consequently, far traveling quarks are suppressed in this framework. The mass function $M_L(p)$ gets strongly suppressed in the infrared when removing the lowest eigenmodes, hence the dynamic mass generation ceases and the bare quark mass is approached successively.

Figure 8: Left: The corrected mass function from the original theory and after having subtracted the lowest 512 Dirac eigenmodes. Right: The same comparison for the wave-function renormalization function of the quark propagator. Courtesy of [252].

Thus, Refs. [253, 256] claim that the rather large mass of hadrons which are built of such truncated quark propagators, cannot be explained by the dynamical $\chi_{SB}$. Reference [256] further shows, that the broken $U(1)_A$ symmetry does not get restored upon unbreaking the chiral symmetry and signals of some higher symmetry that includes chiral symmetry as a subgroup are observed. The classical part of the QCD partition function (the integrand) has, ignoring irrelevant exact zero modes of the Dirac operator, a local $SU(2N_F) \supset SU(N_F)_L \times SU(N_F)_R \times U(1)_A$ symmetry which is absent at the Lagrangian level. This symmetry is broken anomalously and spontaneously. If physics of dynamical $\chi_{SB}$ and of anomaly are encoded in the same near-zero modes, then their truncation on the lattice should recover a hidden classical $SU(2N_F)$ symmetry in correlators and spectra. Recently a global $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_A$ symmetry of the confining Coulombic part of the QCD Hamiltonian has been discovered with $N_F = 2$ [259, 260]. This global symmetry includes both independent rotations of the left- and right-handed quarks in the isospin space as well as the chiral spin rotations that mix the left- and right-handed components of the quark fields. It has been suggested by lattice simulations, however, that a symmetry of mesons in the light quark sector upon the quasi-zero mode truncation from the quark propagators is actually higher than $SU(4)$, because the states from a singlet and a 15-plet irreducible representations of $SU(4)$ are also degenerate. It was demonstrated that classically QCD has a $SU(2N_F)$ symmetry. Emergence of a bilocal $SU(4) \times SU(4)$ symmetry in a Lorentz- and gauge-invariant manner in mesons that contains a global $SU(4)$ as a subgroup upon truncation of the quasi-zero modes. The confining Coulombic part of the QCD Hamiltonian has this bilocal symmetry, which naturally explains a degeneracy of different irreducible representations of $SU(4)$ observed in lattice simulations.
Glozman et al. [261, 262, 263] also reviewed the possibility for existence of cold, dense chirally symmetric matter with confinement. This question can be clarified from spectroscopy of hadrons and their axial properties and the answer crucially depends on the mechanism of mass generation in QCD and interconnection of confinement and $\chi$SB. Almost systematical parity doubling of highly excited hadrons suggests that their mass is not related to $\chi$SB in the vacuum and is approximately chirally symmetric. Then there is a possibility for existence of confining but chirally symmetric matter. A possible mechanism underlying such a phase at low temperatures and large density is the Pauli blocking preventing the gap equation to generate a solution with broken chiral symmetry. However, the chirally symmetric part of the quark Green function as well as all color non-singlet quantities are still infrared divergent, meaning that the system is confined. A possible phase transition to such a matter is most probably of the first order, because there are no chiral partners to the lowest lying hadrons.

Finally, the interplay of confinement and $\chi$SB in QCD is investigated using the Schwinger model in [264]. The dependence of the chiral condensate in the string on the chromo-electric field can be evaluated analytically, and is determined by the chiral anomaly and the $\theta$-vacuum structure. Therefore, the picture allows to predict the distribution of the chiral condensate in the plane transverse to the axis connecting the quark and anti-quark and partial restoration of chiral symmetry in a confining string is found, in good agreement with lattice QCD results.

### 4.4 Dirac Spectral Density and Chiral Polarization

The Banks–Casher relation links the spontaneous $\chi$SB in QCD to the presence of a nonzero density of quark modes at the low end of the spectrum of the Dirac operator. Spectral observables like the number of modes in a given energy interval are renormalizable and can therefore be computed using the Wilson formulation of lattice QCD even though the latter violates chiral symmetry at energies on the order of the inverse lattice spacing [78]. Using numerical simulations, the chiral condensate can be accurately calculated simply by counting the low modes on large lattices and one finds that the low quark modes do condense in the expected way.

Since the Dirac eigenmodes appear in pairs with eigenvalues $\pm \lambda$ and have opposite chiralities, there can be no preference for left- or right-handed modes, hence the modes have to have specific chiral properties locally. While global chirality of Dirac nonzero modes vanishes, the local chiral behavior reflects properties of the underlying gauge background [265, 266].

The simplest “local inquiry” of the above type is whether values $\psi(x) = \psi_L(x) + \psi_R(x)$ tend to appear with equal participation of left/right subspaces or with asymmetric one. In other words, whether they tend to be chirally polarized or anti-polarized. Refs. [267, 268] build upon the X-distribution approach of Ref. [265, 266], considering the left-right decomposition (73) of the local value $\psi_\lambda(x)$ of Dirac modes. For an ensemble of gauge configurations they analyze a probability distribution $P_\lambda(|\psi_1|, |\psi_2|)$ of these local values in some surrounding $\delta \lambda$. In order to determine whether the dynamics of QCD enhances or suppresses the polarization, they define an uncorrelated distribution $P^u_\lambda(|\psi_1|, |\psi_2|) = P_\lambda(|\psi_1|)P_\lambda(|\psi_2|)$ from $P_\lambda(|\psi_2|) = \int d\psi_1 P_\lambda(|\psi_1|, |\psi_2|)$. Then, they determine whether the correlation $C_\lambda$ for a sample chosen from $P^u_\lambda$ is more polarized than a sample chosen from $P_\lambda$, indicating enhanced polarization for $C_\lambda > 0$ and anti-correlation for $C_\lambda < 0$.

The dynamical information associated with the above approach is encoded in the spectral behavior of $C_\lambda \equiv C_\lambda(\lambda)$, with the spectral average in the finite volume defined by [269]

$$C_\lambda(\lambda, M, V) \equiv \frac{\sum_k \langle \delta(\lambda - \lambda_k)C_{A,k} \rangle_{M,V}}{\sum_k \langle \delta(\lambda - \lambda_k) \rangle_{M,V}}$$  \hspace{1cm} (114)
Here $M \equiv (m_1, m_2, \ldots, m_{N_f})$ is the set of quark masses and $C_{A,k}$ the correlation associated with $k$-th mode. In Ref. [267] it was found that $C_{A}(\lambda)$ in quenched QCD has a positive core around zero, and switches to negative values at chiral polarization scale $\Lambda_{ch}$. It was also shown that $\Lambda_{ch}$ is nonzero in the continuum limit at fixed physical volume. Ref. [268] presents evidence that i) $\Lambda_{ch}$ remains positive in the infinite volume limit and is thus a dynamical scale in the theory, ii) $\Lambda_{ch}$ is nonzero in the chiral limit of $N_f = 2 + 1$ QCD and spontaneous $\chi$SB thus proceeds via chirally polarized modes, and iii) $\Lambda_{ch}$ vanishes simultaneously with the density of near-zero modes when temperature is turned on, and is thus a scale closely tied to spontaneous $\chi$SB.

The behavior of $C_{A}(\lambda)$ for an $L = 32a$ lattice with $a = 0.085$ fm of quenched QCD in Fig. 9 (left) shows that the lowest modes exhibit a dynamical tendency for chirality, while the higher modes dynamically suppress it. Chirally polarized low-energy modes condense and are thus carriers of the symmetry breaking. The width $\Lambda_{ch}$ of the band of condensing modes provides a new dynamical scale as the dependence on the infrared cutoff in Fig. 9 (left) indicates, where the numerical data are compared with a fit of the form $\Lambda_{ch}(1/L) = \Lambda_{ch}(0) + b(1/L)^3$ and the cutoff $1/L$ itself. This fit yields an infinite volume limit of $\Lambda_{ch} \approx 160$ MeV. Further, [268] presents evidence that $\Lambda_{ch}$ is nonzero in the chiral limit of $N_f = 2 + 1$ QCD and spontaneous $\chi$SB thus proceeds via chirally polarized modes, and $\Lambda_{ch}$ vanishes simultaneously with the density of near-zero modes when temperature is turned on.

In [270], confinement and $\chi$SB are studied from spectral density of the Dirac operator in $SU(3)$ gauge theories with fundamental quarks. The spectral density $\rho(\lambda)$ is the right derivative of cumulative function $\sigma(\lambda) \equiv \langle \sum_{0 < \lambda_i < \lambda} 1 \rangle / V$, where $V$ is the 4-volume and $\lambda_i$ (real numbers) have magnitudes of Dirac eigenvalues and signs of their imaginary parts. Indicated exclusion of exact zeromodes from counting is harmless since their effect vanishes in the infinite-volume limit. Given the finite statistics of any simulation, coarse-graining of $\rho(\lambda)$ is unavoidable. In the absence of suspicion for singularity the symmetric definition away from origin is used, namely

$$\rho(\lambda, \delta) \equiv \frac{\sigma(\lambda + \delta/2) - \sigma(\lambda - \delta/2)}{\delta} \quad \lambda \geq \delta/2$$

When estimating the infinite-volume value of $\rho(\lambda \to 0)$ from finite systems, it is convenient to work with “right derivative” form, i.e., $\rho(\lambda = 0, \Delta) \equiv \sigma(\Delta) / \Delta$. Valence chiral condensate with overlap is identical to $\pi \lim_{\lambda \to 0} \lim_{V \to \infty} \rho(\lambda, V)$, as is formally in the continuum. The possible behaviors of the spectral density can be represented by three distinct plots, shown in Fig. 10. The monotonic cases are standard and entail confinement with valence $\chi$SB (A) or the lack of both (C,C'). The bimodal (anomalous) option (B) was frequently regarded as an artifact (lattice or other) in previous studies,
but numerical evidence in [270, 271, 272, 273] shows that it persists in the continuum limit, concluding that it informs of a non-confining phase with broken valence chiral symmetry.

4.5 Instanton Liquid Model of Chiral Symmetry Breaking

The results in the last sections lead to the question about the origin of the near-zero modes. A first indication about the origin of the near-zero modes relies on instantons [274, 275, 276, 277] and a well-established theory of chiral symmetry breaking (χSB) came from the instanton liquid model [278, 279, 280, 242, 281]. The solutions to the classical Yang-Mills equations of motion can be classified w.r.t. their topology and different vacua are characterized by a winding number [275]. Instantons and anti-instantons are topological objects which are localized in space-time and allow transitions between neighboring winding numbers (“tunneling”). Hence, they have topological charge $Q = \pm 1$ and according to the Atiyah-Singer index theorem [28, 282, 283, 59] give rise to a single zero mode $\psi_0$ with eigenvalue $\lambda = 0$ and definite chirality, i.e., they exhibit either $\psi_l$ or $\psi_r$, which is concentrated at the instanton core. The instanton solution of topological charge $Q$, localized around $a_\mu$ reads

$$A_\mu(x) = \eta^{i\mu\nu} \sigma_i \frac{(x-a)_\nu}{(x-a)^2 + \rho^2},$$

where $\rho$ is the instanton radius and the ’t Hooft tensor is defined by

$$\eta^{i\mu\nu}(x) = -i\eta_{i\nu\mu} = \begin{cases} e^{i\mu\nu} & \mu, \nu = 1, 2, 3 \\ \pm \delta^{i\mu} & \nu = 4, Q = \pm 1 \end{cases}$$

(117)

It was a great achievement by ’t Hooft to compute the functional determinant $Z$ in the 1-instanton background [278]. To get a finite result, $Z$ has to be normalized to the $A_\mu = 0$ case, regularized and renormalized. The partition function in semiclassical approximation is

$$\frac{Z}{Z_0} = V \int_0^\infty d\rho D(\rho) = V \bar{D},$$

(118)

where $D(\rho)$ is the density of instantons of radius $\rho$. The sum of well-separated instantons is also an approximate solution of the YM equations and the partition function of this so-called instanton gas is

$$Z = \sum_{N=0}^\infty Z_N, \quad Z_N \approx \frac{1}{N!} (V \bar{D})^N$$

(119)

The sum is dominated by instanton configuration with density $N/V = \bar{D}$. Unfortunately $\bar{D}$ is infinite and the assumption of a diluted gas turns out to be wrong. The probability of small size instantons is low because $D(\rho)$ vanishes rapidly for small distances. On the other hand for large distances $D(\rho)$ blows up and soon gets large - this is the origin of the infrared problem. For larger and larger distances, the
vacuum gets more and more filled with instantons of increasing size. At some scale the instanton gas approximation breaks down and one has to consider the interaction between instantons being repulsive to stabilize the medium. The stabilization might occur at distances at which a semiclassical treatment is still possible and at densities at which the various instantons are still well separated objects, \textit{i.e.}, not much deformed through their interaction. So there is a narrow region of allowed values for the instanton radius. This picture of the vacuum is called the instanton liquid model which has been confirmed by different approaches.

The simplest suggestion is to introduce a cutoff $\rho_c$ and to ignore large instantons \cite{284,285}

$$D_{\rho_c} = \int_0^{\rho_c} d\rho D(\rho). \quad (120)$$

The cutoff has to be chosen sufficiently small such that the space-time fraction $f$ filled with instantons is smaller than 1 in order to justify the model of a diluted gas.

$$f = \frac{2}{N_c} \int_0^{\rho_c} d\rho \frac{1}{2} \pi^2 \rho^4 D(\rho) < 1 \quad (121)$$

This simple cutoff procedure can be improved by introducing a scale invariant (hardcore) repulsion between instantons, which effectively suppresses large instantons \cite{278}. This procedure has the advantage of respecting the scaling Ward identities which are otherwise violated by the simple cutoff ansatz. In \cite{286,287} such a repulsion has been found leading to a phenomenologically favorable packing fraction. Unfortunately this repulsion is an artifact of the sum-ansatz as has been shown by \cite{288}. Therefore the infrared problem was still unsolved but it was possible to make successful prediction by simply assuming a certain instanton density and some average radius. It seems that the vacuum can be described by effectively independent instantons of size $\rho = 600$ MeV$^{-1}$ and mean distance $L_0 = 200$ MeV when the integral instanton density is fixed by the gluon condensate \cite{289}:

$$n = N/V = 1/L_0^4 = \frac{g^2}{32\pi^2} < F_{\mu\nu}^a F_{\mu\nu}^a> = (200 \text{MeV})^4. \quad (122)$$

The Instanton Liquid Model defined by this assumption very successfully describes the physics of light hadrons, see e.g., \cite{290,291,292,293,294,295,242} for extensive numerical studies. In high energy processes with momentum transfer $p$ of $1-10$ GeV, $D(\rho)$ is often multiplied with function, which is sharply peaked around $\rho \sim p^{-1}$. The integral over $\rho$ is then dominated by small instantons and is infrared convergent. The results are, hence, independent of the infrared cutoff and no additional assumptions have to be made. A similar phenomenon to derive a relation between the quark condensate and the QCD scale $\Lambda$ without model assumptions, in agreement with experiment.

In the instanton liquid model overlapping would-be zero modes split into low-lying nonzero modes which create the chiral condensate \cite{278,279,296}. For field configurations with instantons and anti-instantons the (would-be) zero modes get small shifts of their eigenvalues and distribute around zero along the imaginary axis as the Dirac operator is anti-Hermitian, they become near-zero modes. Hence, overlapping would-be zero modes belonging to single instantons or anti-instantons split into low-lying nonzero modes and contribute to the density of near-zero modes, breaking chiral symmetry via Banks-Casher.

The instanton liquid model provides a physical picture of $\chi_{SB}$ by the idea of quarks “hopping” between random instantons and anti-instantons, changing their helicity each time. This process can be described by quarks propagating between quark-instanton vertices. In the random instanton ensemble one finds the value of the chiral condensate $\langle 0|\bar{\psi}\psi|0 \rangle \approx -(253 \text{ MeV})^3$ \cite{297}, which is quite close to the lattice values in Eq. (103) and only depends on a scale given by the average instanton size. Despite their striking success providing a mechanism for $\chi_{SB}$, instantons are not
able to explain quark confinement. There are models where instantons may split into dyons or pairs of magnetic monopoles, which may provide a monopole-like confinement mechanism via the Dual Superconductor scenario. The classical solutions are called merons [298, 299, 300, 301, 302, 303, 304, 305], bions [306, 307, 308, 309, 310, 311, 312, 313] or at finite temperature calorons [314, 315, 316, 317, 318, 319, 320, 321, 322].

A meron can be viewed as a tunneling event between two Gribov vacua. In that picture, the meron is an event which starts from vacuum, then a Wu-Yang monopole emerges, which then disappears again to leave the vacuum in another Gribov copy. In contrast to instantons, the tunneling process described by a single meron changes the sign of the Polyakov loop $P$. Furthermore, the asymptotic values of $P$ for a meron–anti-meron pair and a meron-meron pair are of opposite sign. Thus, merons do not single out either one of the two center-elements of SU(2) and have the potential of generating a center symmetric ensemble. Magnetic bion condensation provides new mechanism of confinement and mass gap in QCD(adj) formulated on small $S^1 \times R^3$ based on symmetries, an index theorem, and Abelian duality. If the magnetic charge of the BPS monopole is normalized to unity, Ünsal shows that confinement occurs due to condensation of objects with magnetic charge 2, not 1 [312]. Because of index theorems, such an object cannot be a configuration of two identical monopoles. Its net topological charge must vanish, and hence it must be topologically indistinguishable from the perturbative vacuum. Bions provide such non-self-dual topological excitations, constructed from magnetically charged, topologically null molecules of a BPS monopole and $\overline{K}\bar{K}$ anti-monopole. Bion condensation is also the mechanism of confinement in $\mathcal{N} = 1$ SYM on the same four-manifold and the $SU(N)$ generalization hints a possible hidden integrability behind non-supersymmetric QCD of affine Toda type, and allows analytical computation of the mass gap in the gauge sector.

Calorons are the instanton solutions by Kraan/van Baal and Lee/Lu (KvBLL) at finite temperature. In other words, their base space is $R^3 \times S^1$ where the circle $S^1$ has circumference $\beta = 1/T$ as usual. As it turns out from the explicit solutions [314, 317, 319], calorons consist of localized lumps of topological charge density, which – due to self-duality – are lumps of action density, too. These lumps arise from a non-vanishing $A_0$ at spatial infinity leading to a non-trivial holonomy, the limit of the (untraced) Polyakov loop at spatial infinity, playing the role of a Higgs field fixing a color direction and thus breaking center symmetry by generation of mass for the gauge field along the color $N$-sphere and for the scalar fields perpendicular to that sphere, while the gauge field of the unbroken $U(1)$ symmetry remains massless. In $SU(2)$, the holonomy can be parametrized as $\mathcal{P}_\omega = \exp(2\pi i \sigma_3 \omega/2)$ with $\omega \in [0,1/2]$ and $2\omega$ and $1 - 2\omega$ defining the weights, or ‘masses’ of the caloron constituents. Fig. [11] shows the action density of calorons for various holonomy parameters (top) and instanton radii $\rho$. The extreme cases $\omega = 0, 1/2$ give trivial holonomy $\mathcal{P}_\omega = \pm \mathbb{1}_2$ and thus no symmetry breaking (the Higgs field vanishes), that is the so-called HS caloron [314] with only one magnetic monopole. For the symmetric case $\omega = \tilde{\omega} = 1/4$ both monopoles have equal mass and are identical from the point of view of action density. The holonomy is on the equator of $S^3$, which makes this case attractive for the confined phase. When close together the constituents develop a time-dependence and merge to an instanton-like lump of size $\rho$. In the regime of well-separated monopoles, the action density becomes static and the distance takes over the meaning of the caloron’s size.

For the gauge group $SU(N)$ one can have up to $N$ lumps per unit topological charge, realizing the old idea of ‘instanton quarks’ [323] of fractional charge. They possess (quantized) magnetic charge equal to their electric charge and hence are called dyons, which combine to magnetic neutrality within a caloron, however. The moduli of calorons are the spatial locations of the dyons, which can take any value, plus phases [316]. Again, when well separated, these lumps are static, the gauge field, generically, is time dependent however, as the important (super-selection) parameter is the holonomy. Under the conjecture that the asymptotic Polyakov loop is related to the average Polyakov loop, the order parameter of confinement, calorons are sensitive to the phases of QCD.
Figure 11: Plots of the action density of various calorons as a function of two spatial directions, show the influence of the holonomy parameter $\omega$ (top, from left to right $1/4$, $1/8$ and $0$) on the monopole masses and of the size parameter $\rho$ (bottom, from left to right $1.6$, $1.2$ and $0.8$ in units of $\beta$, $\omega = 1/8$) on the separation of the monopoles. Courtesy of [317].

Recently, extensive studies of the gauge topology using the instanton-dyon liquid model have been performed in a series of papers [324, 325, 326, 327, 328, 329, 330, 331, 332, 333]. The starting point of the model are the KvBLL instantons (calorons) threaded by finite holonomies and their splitting into instanton-dyon constituents, with strong semi-classical interactions. At low temperature, the phase preserves center symmetry but breaks chiral symmetry spontaneously. At sufficiently high temperature, the phase restores both symmetries as the constituent instanton-dyons regroup into topologically neutral instanton–anti-instanton molecules. Thus, the instanton-dyon liquid model unifies instanton and monopole features, i.e., $\chi_{SB}$ and confinement properties. Further investigations of instantons and monopoles are found in [334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345]. The relation of calorons and center vortices, which will be considered in the next section, was studied in [346].

4.6 Center Vortex Model of the QCD Vacuum

Center vortices [211, 347, 348, 349, 350, 351] are promising candidates for explaining confinement. They form closed magnetic flux tubes, whose flux is quantized, taking only values in the center of the gauge group. These properties are the key ingredients in the vortex model of confinement, which is theoretically appealing and was also confirmed by a multitude of numerical calculations, both in lattice Yang-Mills theory and within a corresponding infrared effective model, see e.g., [352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364], or [218], which summarizes the main features.

Vortices that randomly penetrate a given Wilson loop very naturally give rise to an area law. Since vortices are closed surfaces, the necessary randomness can be facilitated only by large vortices. This is further translated into the percolation of vortices, meaning that the size of the (largest) vortex clusters becomes comparable to the extension of the space itself. This percolation has been observed in lattice simulations of the confined phase, see e.g., [353], while in the deconfined phase the vortices align in the time-like direction and the percolation mechanism remains working only for spatial Wilson loops [355, 358]. This parallels percolation properties of monopoles. Moreover, it conforms with the observation at high temperatures that the spatial Wilson loops keep a string tension in contrast to the correlators of Polyakov loops.
Due to the color screening by gluons the string tension of pairs of static color charges in $SU(N)$ gauge theories depends on their $N$-ality. From the field perspective this $N$-ality dependence has its origin in the gauge field configurations which dominate the path integrals in the infrared. Center vortices are the only known configurations with appropriate properties. Recent results [365] have also suggested that the center vortex model of confinement is more consistent with lattice results than other currently available models. If one considers that a phase transition of the gauge field influences both gluons and fermions, then one would expect that deconfinement and chiral phase transition are directly related and rely on the same mechanism.

Lattice studies indicate that vortices are also responsible for topological charge [366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377] and $\chi$SB [378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390], and thus unify all non-perturbative phenomena engendered by the structure of the strong interaction vacuum in a common framework.

A similar picture to the instanton liquid model exists insofar as lumps of topological charge arise at the intersection and writhing points of vortices. In [380, 391] it was shown that center vortices, quantized magnetic fluxes in the QCD vacuum, contribute to the topological charge by intersections with $Q_U = \pm 1/2$ and writhing points with a value of $\pm 1/16$. By the Atiyah-Singer index theorem [28, 282, 283, 59] zero modes are related to one unit of topological charge. Therefore, the question emerges, how vortex intersections and writhing points are related to these zero modes. Ref. [372] compares vortex intersections with the distribution of zero modes of the Dirac operator in the fundamental and adjoint representation using both the overlap and asqtad staggered fermion formulations in SU(2) lattice gauge theory. By forming arbitrary linear combinations of zero modes they prove that their scalar density peaks at least at two intersection points [372]. Further, since it is expected that zero modes of the Dirac operator concentrate in regions of large topological charge density, a correlation between the location of vortex intersections and writhing points and the density $\rho_\lambda(x) = \langle \lambda | x \rangle \langle x | \lambda \rangle$ of eigenmodes of the Dirac operator $D$, where $D|\lambda\rangle = \lambda|\lambda\rangle$ with $\lambda = 0$ in the overlap formulation and $\lambda \approx 0$ in the asqtad staggered formulation supports this picture [385]. Ref. [392] proposed the observable

$$C_\lambda(N_v) = \frac{\sum_{p_i} \sum_{x \in H} (V \rho_\lambda(x) - 1)}{\sum_{p_i} \sum_{x \in H} 1},$$

(123)

as a measure for the vortex-eigenmode correlation. To explain this formula one has to recall that center vortices are located by center projection in maximal center gauge [393]. Plaquettes on the projected lattice, “P-plaquettes”, are either $+1$ or $-1$; they form closed surfaces on the dual lattice. Each point on the vortex surface belongs to $N_v$ P-plaquettes. $N_v = 0$ means points which do not belong to a vortex surface, $N_v = 1$ or 2 is impossible since vortex surfaces are closed, for corner points or points where the surface is flat one get $N_v = 3, 4, 5$, when the surface twists around a point $N_v = 6$ or 7 and at points where surfaces intersect $N_v \geq 8$. In Fig. 12 (left), the data for $C_\lambda(N_v)$ versus $N_v$ computed for eigenmodes of the overlap Dirac operator is displayed. The lattice configurations are generated by Monte-Carlo simulations of the Lüscher-Weisz action at $\beta_{\text{LW}} = 3.3$. The correlations for the first eigenmode and the twentieth Dirac eigenmode are shown. Since the correlator increases steadily with increasing $N_v$, hence the Dirac eigenmode density is significantly enhanced in regions of large $N_v$. The correlation is strong for zero- and low-lying modes, and decreases for higher eigenmodes. The inverse participation ratio for the normalized eigenmode density $\rho_\lambda(x)$

$$IPR = N \sum_x \rho_\lambda^2(x), \quad \text{with} \quad \sum_x \rho_\lambda(x) = 1$$

(124)

should scale with lattice spacing as $1/a^{4-d}$ if the eigenmode has support mainly on a submanifold of dimension $d$. Asqtad staggered eigenmodes on center-projected configurations have support on point- and surface-like regions, as shown in Fig. 12 (right). The authors of [392] further conclude
that the eigenmode correlation on two-dimensional surfaces (vortices) is stronger than for three-dimensional objects. A positive low-lying Dirac eigenmode - vortex structure resp. topological charge correlation is a first indication of the importance of center vortices for $\chi_{SB}$.

Another remarkable result was found in [378], they remove center vortices from an ensemble of lattice SU(2) gauge configurations, which adds short-range disorder. Nevertheless, they observe long-range order in the modified ensemble: confinement is lost and chiral symmetry is restored (together with trivial topology), proving that center vortices are responsible for both phenomena.

A similar approach was investigated in [394], studying the interplay between Dirac eigenmodes and center vortices in SU(2) lattice gauge theory. The authors focus on vortex-removed configurations and compare them to an ensemble of configurations with random changes of the link variables. They show that removing the vortices destroys all zero modes and the near-zero modes are no longer coupled to topological structures. The Dirac spectrum for vortex-removed configurations in many respects resembles a free spectrum thus leading to a vanishing chiral condensate. Configurations with random changes leave the topological features of the Dirac eigensystem intact, see Fig. 13 (left). They also show that smooth center vortex configurations give rise to zero modes and topological near-zero modes, while in [385] this was confirmed also on vortex-only configurations. In particular, it was shown that center-projected SU(2) lattice configurations give rise to a dense low-lying Dirac eigenvalue spectrum, as required for $\chi_{SB}$, for a massless lattice Dirac operator in the asqtad staggered formulation, see Fig. 13 (right).

In contrast, this low-lying spectrum is not found for chirally improved and overlap Dirac operators on center-projected lattices, for reasons which are almost certainly connected to the lack of smoothness of center-projected configurations. Chiral symmetry is absent in the chirally-improved Dirac operator on such configurations, while for the overlap operator an exact symmetry is present, but is strongly field-dependent for rough configurations (and thus quite different from the continuum symmetry). In the case of the overlap operator we have found that a moderate degree of smoothing of the center-projected lattice brings back the low-lying spectrum. In a staggered formulation such as asqtad, the smoothness of the lattice configuration has nothing to do with the exact chiral symmetry, and the low-lying modes are present for thin vortex configurations.

In recent investigations further sources of topological charge from center vortices were discovered. Colorful spherical SU(2) vortices [369] [371] [373] [374] [387] and colorful plain vortices [376] [395]
Figure 13: (left) Chirally-improved Dirac eigenvalues $\lambda$ in the complex plane. The random (sign) changes of the gauge field, as well as the procedure of vortex-removal, both started from the same original gauge configuration - the used for the center plot. Random changes have almost no effect while vortex removal restores chiral symmetry. Courtesy of [394]. (right) The first twenty asqtad staggered Dirac eigenvalue pairs from 200 configurations on a $16^4$ lattice, generated by Monte-Carlo simulation of the tadpole improved Lüscher-Weisz pure-gauge action, at coupling $\beta_{LW} = 3.3$ (lattice spacing $a = 0.15$ fm) for the $SU(2)$ gauge group. Courtesy of [385].

were introduced. They contribute to the topological charge by their color structure and attract zeros mode like instantons. A spherical vortex can be constructed in one time-slice $t_0$, by putting a hedgehog-like gauge field on e.g., the time-like links $U_4(\vec{r}, t_0) = e^{i\alpha(\vec{r})\vec{n}(\vec{r})\vec{\sigma}} \in SU(2)$ around a two-dimensional sphere in $R^3$. $\alpha(r)$ varies monotonously between 0 and $\pi$ from the center $r = 0$ of the sphere to large distances and the color direction is chosen as $\vec{n} = \vec{r}/r$. As the field at large distances is independent of the direction this ball in $R^3$ is isomorphic to $S^3 \cong SU(2)$ and characterized by a winding number $N = -1/24\pi^2 \int d^3 r \varepsilon_{ijk} \text{Tr}[(U_4^{\dagger} \partial_i U_4)(U_4^{\dagger} \partial_j U_4)(U_4^{\dagger} \partial_k U_4)]$, resulting in $N = \pm 1$ for the spherical vortices and accordingly in a topological charge via the Atiyah-Singer index theorem. In [373], the continuum object corresponding to the spherical vortex was identified by a gauge transformation transferring the topological structure from the time-like links to the corresponding space-like links. After this gauge transformation the spherical vortex can be distributed over several time slices and was identified as vacuum to vacuum transition in temporal direction and its similarity to an instanton was demonstrated. Similarly color structures were introduced in [376, 395] on plain vortex structures.

Further, in [387], it was shown how the interplay of various topological structures from center vortices (and instantons) leads to near-zero modes, which by the Banks-Casher relation [43] are responsible for a finite chiral condensate, using the overlap and asqtad staggered Dirac operators. As an example, the spectra of (anti-/instantons, spherical (anti-/vortices and pairs of two of these individual objects on otherwise trivial lattice configurations are shown in Fig. [14]. One can see nearly exactly the same eigenvalues for instantons and spherical vortices, as well as different pairs, single objects attract a zero mode whereas would-be zero modes of two objects in one lattice configuration result in a pair of near-zero modes, consisting of two chiral parts corresponding to the two
constituents of the pairs. Similar results were presented for vortex intersections from plane vortices, which due to their topological charge contributions give rise of zero and near-zero modes [372].

These observations lead to a picture similar to the instanton liquid model. The lumps of topological charge appearing in Monte-Carlo configurations interact in the QCD-vacuum and determine the density of near-zero modes. Therefore, it is not the true zero modes deciding on the value of the topological charge of a field configuration which lead to $\chi_{SB}$. The number of these modes is small in the continuum limit. It is the density of interacting topological objects which leads to the density of modes around zero and according to the Banks-Casher relation determines the strength of $\chi_{SB}$.

The instanton liquid model provides a physical picture of $\chi_{SB}$ via the idea of quarks hopping between random instantons and anti-instantons, changing their helicity each time. This process can be described by quarks propagating between quark-instanton vertices. As fermions do not seem to make much of a difference between instantons and spherical vortices this picture can be extended to colorful spherical center vortices. In fact, the spherical vortices reproduce all characteristic properties of low-lying modes necessary for $\chi_{SB}$ [265]:

- Their probability density is clearly peaked at the location of the vortices.
- The local chirality at the peaks exactly matches the sign of the topological charge and the size of the chiral lump is correlated to the extension of the topological structure.
- As a spherical vortex and an anti-vortex approach each other, the eigenvalues are shifted further away from zero and the localization and local chirality properties fade.

In the vortex picture the model of $\chi_{SB}$ can be formulated even more generally, as it was shown that various shapes of vortices attract (would-be) zero modes which contribute via interactions to a finite density of near-zero modes with local chiral properties, i.e., local chirality peaks at corresponding topological charge contributions. In Monte-Carlo configurations, there are no perfectly flat or spherical vortices, as one does not find perfect instantons, but the general picture of topological charge from vortex intersections, writhing points and even color structure contributions or instantons can provide a general picture of $\chi_{SB}$: any source of topological charge can attract (would-be) zero modes and produce a finite density of near-zero modes leading to $\chi_{SB}$ via the Banks-Casher
relation. Here one also has to ask what could be the dynamical explanation of $\chi_{SB}$. One can try the conjecture that only a combination of color electric and magnetic fields leads to $\chi_{SB}$, electric fields accelerating color charges and magnetic fields trying permanently to reverse the momentum directions on spiral shaped paths. Such reversals of momentum keeping the spin of the particles should especially happen for very slowly moving color charges. Alternatively one could argue that magnetic color charges are able to flip the spin of slow quarks, i.e., when they interact long enough with the vortex structures.

Therefore, it seems that vortices not only confine quarks into bound states, but also change their helicity in analogy to the instanton liquid model. While there is no conclusive answer to the question of a dynamical explanation for the mechanism of $\chi_{SB}$, one can speculate that the generation of near-zero modes demonstrated for vortex and instanton configurations, carries over to vortices present in Monte-Carlo generated configurations. As the near-zero modes are located around intersection and writhing points of vortices that carry topological charge, the behavior away from these points would seem to be far less important. Other mechanisms of $\chi_{SB}$, in addition to the instanton liquid paradigm or the vortex picture may be operative in the Yang-Mills vacuum. For instance, it also seems possible that, even in the absence of would-be zero modes, the random interactions of quarks with the vortex background may be strong enough to smear the free dispersion relation such that a finite Dirac operator spectral density at zero virtuality is generated. In fact, a confining interaction by itself generates $\chi_{SB}$, independent of any particular consideration of would-be zero modes connected to topological charge [212]. However, this effect on its own is not sufficiently strong for a quantitative explanation of the chiral condensate; other effects, among them possibly the ones considered above, must play a role.

The Adelaide group [396, 386, 389, 390, 397, 398, 399, 400] shows that center vortices underpin both, confinement and dynamical $\chi_{SB}$ in $SU(3)$ lattice gauge theory. They look at the topological charge density, the static quark potential, the quark mass function, and the hadron mass spectrum on original (untouched), vortex-only and vortex-removed ensembles. In a covariant gauge, the lattice quark propagator in momentum space can be decomposed into Dirac scalar and vector components as

$$ S(p) = \frac{Z^R(p)}{iq + M(p)}, $$

(125)

where $M(p)$ is the non-perturbative mass function, $Z^R(p)$ containing all renormalization information and $q$ is the kinematic lattice momentum. The infrared behavior of the Landau-gauge quark propagator has been used as a probe of dynamical $\chi_{SB}$ [401, 402, 403, 404]. At low momenta, enhancement of the mass function $M(p)$ provides a clear signal of dynamical mass generation, and thus of dynamical $\chi_{SB}$. In $SU(2)$ gauge theory, the mass function clearly displays the absence of dynamical $\chi_{SB}$ upon center vortex removal, as the mass function does not develop a dynamically generated mass in the infrared limit [397]. A similar study in $SU(3)$ gauge theory using the asqtad staggered fermions did not reveal a comparable role in dynamical $\chi_{SB}$ [386], the mass function sustained dynamical mass generation on vortex-removed configurations.

However, in Ref. [396], where the vortex-removed hadron spectrum was studied with Wilson fermions, it became clear that this residual mass generation on vortex-removed configurations was not associated with chiral symmetry, i.e., chiral symmetry was indeed lost upon vortex removal. Both the asqtad staggered and Wilson fermion actions explicitly break chiral symmetry, and hence the relation between center vortices and dynamical $\chi_{SB}$ may be clouded by the resulting lattice artifacts. In [390], the Landau-gauge overlap quark propagator was used, results on original (untouched) and vortex-removed ensembles are plotted in Fig. [15]. The renormalization function shows similar behavior in both cases, however, the mass function reveals a significant change upon vortex removal. Unlike the asqtad staggered propagator, which showed little to no change in the infrared enhancement [386], the overlap operator is able to reveal the subtle damage caused to the gauge
fields through vortex removal. The removal of the vortex structure from gauge fields spoils dynamical mass generation, and thus dynamical $\chi_{SB}$.

![Graphs showing mass and renormalization functions](image)

Figure 15: The mass (left) and renormalization (right) functions on the original (untouched) (squares) and vortex-removed (crosses) configurations. Removal of the vortex structure from the gauge fields spoils dynamical mass generation and thus $\chi_{SB}$. Courtesy of [390].

As mentioned above, the overlap fermion action is not well defined on vortex-only configurations due to their lack of smoothness. Therefore, to quantify the extent to which the center-vortex information can give rise to dynamical mass generation, the overlap quark propagator on both untouched and vortex-only configurations is calculated after 10 sweeps of cooling. As expected, the mass function on the original configurations shows some reduction in dynamical mass generation, while being qualitatively similar to the uncooled results [405]. It is interesting however, that the vortex-only results for the mass function are strikingly similar to the untouched, as illustrated in Fig. 16 (left). The renormalization functions also share a similar behavior, hence the vortex-only configurations reproduce almost all dynamical mass generation.

Further, the background of instanton-like objects emerging from the vortex-only configurations under cooling is examined in [399] by examining the local maxima of the action density. It is shown that after just 10 sweeps of smoothing the local maxima stabilize and begin to resemble classical instantons in shape and corresponding topological charge density at the center [406]. The average number of these maxima per configuration during cooling is plotted in Fig. 16 (right) for up to 200 sweeps. The number of instanton-like objects found on original and vortex-only configurations remains about equal even after large amounts of cooling, whereas the number of objects on vortex-removed configurations is greatly reduced. Thus, while vortex-removal destabilizes the otherwise topologically non-trivial instanton-like objects it is possible to create an instanton liquid-like background on vortex-only configurations, analogous to that found on Monte-Carlo generated configurations after similar smoothing. Through calculations of the static quark potential and Landau-gauge overlap propagator, it was shown that this background is able to reproduce all salient long-range features of the original configurations. Therefore, the information necessary to recreate the long-range structure of the QCD vacuum is contained within the center vortex degrees of freedom and the presented results in this section in particular provide evidence that the center vortex structure of the vacuum plays a fundamental role in dynamical $\chi_{SB}$ in $SU(2)$ and $SU(3)$ gauge theory.
Figure 16: (left) The mass function on the original (untouched) (squares) and vortex-only (circles) configurations after 10 sweeps of three-loop $\mathcal{O}(a^4)$-improved cooling, at an input bare quark mass of 12 MeV. (right) A log plot of the number of instanton-like objects per configuration (ensemble-average number of local maxima of the action) found on untouched, vortex-only and vortex-removed ensembles as a function of $\mathcal{O}(a^4)$-improved cooling sweeps. Courtesy of [390].

4.7 Dimensionality Investigations and Center Domains

The importance of the long-range nature of low-dimensional topological structures for the understanding of the mechanism of $\chi$SB in QCD was also underlined by Buividovich et. al [407, 408, 417, 409] and agrees well with a vortex picture of $\chi$SB. Since the QCD-vacuum is strongly non-perturbative, it does not contain semiclassical instantons [265, 266] but is crowded with topologically charged objects which, after smooth reduction of the action (also known as cooling), may become instantons. In pure $SU(3)$ lattice gauge theory in a typical equilibrium configuration about 80% of space-time points are covered by two oppositely-charged connected structures built of elementary three-dimensional coherent hypercubes. The hypercubes within the structure are connected through two-dimensional common faces suggesting that this coherence is a manifestation of a low-dimensional order present in the QCD vacuum [407, 408].

Ref. [409] demonstrates that the above mentioned smoothing procedures affect the dimensionality of the regions where the topological charge density $q(x)$ is localized. They measure the local density $q(x)$ of the topological charge with the trace of the zero-mass Neuberger operator $D$ [35, 37]

$$q(x) = -\frac{1}{N_f} \text{Tr} \left[ \gamma_5 \left( 1 - \frac{a}{2} D(x,x) \right) \right],$$

where the trace is taken over spinor and color indices. These investigations demonstrate that topological charge and zero modes are localized on low-dimensional fractal structures and tend to occupy a vanishing volume in the continuum limit. With the inverse participation ratio (124) they derive a fractal dimension of fermionic zero modes. Distributions localized on a single site get $IPR = N$ and constant distributions $IPR = 1$. With the eigenfunctions $|\lambda\rangle$ of the overlap Dirac operator to the eigenvalues $\lambda$ they measure the average over all zero modes and all measured gauge field configurations of the local "chiral condensate" and "chirality", or scalar and chiral densities of the eigenmodes

$$\rho_\lambda(x) = \langle \lambda | x \rangle \langle x | \lambda \rangle, \quad \rho_\lambda^5(x) = \langle \lambda | x \rangle \gamma^5 \langle x | \lambda \rangle.$$

(127)
Figure 17: Ordinary IPR for zero modes, Eq. (127). (right) Fractal dimension of topological structures at various cooling stages. The solid line is shown to guide the eye. Courtesy of [409].

The left plot in Fig. 17 shows how the localization depends on the lattice spacing \(a\) and the number of cooling steps. The finer the lattice is the larger gets the IPR. This agrees very well with the idea that the volume occupied by the fermionic zero modes in the continuum limit approaches zero [411]. Since zero modes, \(\lambda = 0\), have definite chirality the results for the local chirality agree with the local chiral condensate.

Performing a number of measurements with various lattice spacings \(a\) Ref. [409] defines a fractal dimension \(d\) via \(\text{IPR}(a) = \text{const}/a^d\), shown in Fig. 17 (right). These results show that fermionic zero modes and chirality are localized on structures with fractal dimension \(2 \leq d \leq 3\), favoring the vortex/domain-wall nature of the localization [380, 412, 413]. The fractal dimension of these structures depends on the number of cooling steps. A long sequence of cooling iterations destroys the low-dimensional structures leading to gauge fields close to classical minima of the action where instantons dominate the properties of field configurations.

In finite temperature \(SU(3)\) lattice gauge theory using the fixed scale approach the authors of [414, 415, 416, 417] study properties of static quark sources represented by local Polyakov loops \(L(x)\). They construct spatial clusters of points \(x\) where the phase of \(L(x)\) have similar values in the vicinity of the center values \(0, \pm 2\pi/3\) and find that below the deconfinement transition the clusters form objects with a fractal dimension \(d < 3\). As the temperature is increased the largest cluster starts to percolate, see Fig. 18, and its dimensionality approaches \(d = 3\).

From the cluster properties a simple qualitative picture for confinement and the deconfinement transition emerges. Below \(T_c\) the clusters of lattice points which have the same center phase information are small. Only if a quark- and an anti-quark source are sufficiently close to each other they fit into the same cluster and can have a non-vanishing expectation value. Sources at distances larger than a typical cluster size receive the independent center fluctuations from different clusters and the correlator averages to zero. Above \(T_c\) the clusters percolate and coherent center information is available also for larger distances allowing for non-vanishing correlation at large separation of the sources. In this picture deconfinement is a direct consequence of a percolating center cluster. The fractal structure of the clusters in the transition region may have implications regarding both the small shear viscosity and the large opacity of the Quark-Gluon Plasma observed in heavy-ion collision experiments.

In [418] the phase transition is probed through visualizations of the Polyakov loop providing novel insights into the structure and evolution of center clusters. Using the HMC algorithm the percolation during the deconfinement transition is observed. Using 3D rendering of the phase and magnitude
of the Polyakov loop, the fractal structure and correlations are examined. The evolution of the center clusters as the gauge fields thermalize from below the critical temperature to above it are also exposed. Using stout-link smearing to remove small-scale noise allows to observe the large-scale evolution of the center clusters. A correlation between the magnitude of the Polyakov loop and the proximity of its phase to one of the center phases of SU(3) is evident in the visualizations.

In full QCD, the fermion determinant describing the dynamical quarks can be expressed as a sum over closed loops, which may be viewed as generalized Polyakov loops and are sensitive to the center properties of the gauge fields [419]. The fermion determinant breaks the center symmetry explicitly and acts like an external magnetic field which favors the real sector (phase 0) for the Polyakov loop $P$. However, preliminary numerical results with dynamical fermions show that locally also the two complex sectors (phases $\pm 2\pi/3$) remain populated. The corresponding clusters will again lead to a coherent phase information for sufficiently close quark lines. However, the explicit symmetry breaking through the determinant leads to a crossover type of behavior in the dynamical case. Further, also the chiral transition may be characterized as a percolation phenomenon as suggested in [420], where the critical behavior of the clusters that arise in the application of the Meron algorithm to a fermion model in 2+1 dimensions was examined. The analysis suggests the occurrence of a generalized percolating phase transition at the chiral critical temperature in close analogy with Fortuin-Kasteleyn percolation in spin models.

4.8 Chiral Magnetic Effect and Vacuum alignment

Chiral symmetry breaking ($\chi$SB) is predicted to be enhanced by the presence of a magnetic field, a phenomenon which is known as magnetic catalysis, nicely reviewed in [421]. It was stated that the quark condensate rises with the increase of the external magnetic field and that the spins of quarks align parallel to the field (magnetization of the QCD vacuum). Numerical simulations in [422, 423, 424, 179, 180, 177, 178, 181, 182, 185, 184] confirm these predictions from chiral perturbation theory for full QCD, it was stressed that the chiral condensate depends linearly on the strength of the applied field, see Fig. [19] and there is a paramagnetic polarization of the vacuum in SU(2) Yang-Mills theory on the lattice. The corresponding susceptibility and other magnetic properties are calculated and compared with the theoretical estimations.
There are nonzero local fluctuations of the chirality and electromagnetic current, which grow with the magnetic field strength. These fluctuations can be a manifestation of the Chiral Magnetic Effect (CME), a CP-odd generation of the electric current of quarks directed along the magnetic field. The near-zero eigenmodes of the Dirac operator tend to have more regular structure with larger (compared to zero-field case) Hausdorff dimensionality, suggesting that the delocalization of near-zero eigenmodes plays a crucial role in the enhancement of the $\chi$SB. The external magnetic field seems to force the quark to develop a local electric dipole moment along the direction of the field and further enhances the presence of the string-like defects which are parallel to the magnetic field, leading to a vortex liquid, electrically superconducting phase [425, 426]. The CME is only one representative of a whole class of anomaly related transport phenomena like Axial Magnetic and Chiral Vortical Effect [427, 428].

Figure 19: (left) The chiral condensate vs. the magnetic field $qB$ at two different temperatures $T$. The solid lines are linear fits with the function $\Sigma(B) = \Sigma_0 (1 + e B / \Lambda^2_B)$, where $\Sigma_0$ and $\Lambda_B$ are the fitting parameters. Courtesy of [177]. (right) The continuum extrapolated lattice results for the change of the condensate as a function of $B$, at six different temperatures. Courtesy of [184].

In [423], the authors report on a mean-field study of spontaneous $\chi$SB for Dirac fermions with contact interactions in the presence of chiral imbalance, which is modelled by nonzero chiral chemical potential. They point out that chiral imbalance lowers the vacuum energy of Dirac fermions, which leads to the increase of the renormalized chiral chemical potential upon $\chi$SB. The critical coupling strength for the transition to the broken phase is slightly lowered as the chiral chemical potential is increased, and the transition itself becomes milder. Furthermore, they study the chiral magnetic conductivity in different phases and find that it grows both in the perturbative weak-coupling regime and in the strongly coupled phase with broken chiral symmetry. In the strong coupling regime the chiral magnetic effect is saturated by vector-like bound states (vector mesons) with mixed transverse polarizations. Similar and further results for magnetic catalysis and electromagnetic background fields were reported in [429, 430, 431, 432, 433, 434] and vacuum alignment was also observed in a different approach:

Confinement in asymptotically free gauge theories is accompanied by the spontaneous breaking of the global flavor symmetry. When a subgroup of the flavor symmetry group of a gauge theory is weakly coupled to additional gauge fields, the vacuum tends to align such that the gauged subgroup is unbroken. Independently, a lattice discretization of the continuum theory typically reduces the manifest flavor symmetry, and, in addition, can give rise to new lattice-artifact phases with sponta-
neously broken symmetries. The authors of [435, 436, 437] discuss the interplay of these two phe-
omena, using chiral Lagrangian techniques in two-flavor Wilson QCD coupled to electromagnetism
and theories with staggered fermions. They show how lattice artifacts, quark-mass induced contrib-
utions, and weak interactions can all compete in determining the pattern of symmetry breaking in
a strongly coupled gauge theory. Of course, in the continuum limit, quark-mass and weak-coupling
effects dominate, but these examples demonstrate that lattice artifacts can mask the correct phase
diagram of the theory. They find that the mechanism of vacuum alignment prevents a condensate
that would imply a dynamical Higgs mechanism in the weak sector from developing.

4.9 Green’s Functions and Topological Configurations

There are two views on the long-distance dynamics of Yang-Mills theory (and QCD), which are cur-ently employed prominently [438]. One view is based on topological configurations of various types
[218], as reviewed in the last few sections. These configurations provide direct access to the gener-
ation of (quark) confinement, chiral symmetry breaking ($\chi_{\text{SB}}$), the finite temperature phase transition
and other non-perturbative features of Yang-Mills theory and QCD. The other view is by means of
Green’s functions [404]. These provide access to (gluon) confinement, $\chi_{\text{SB}}$, bound states and also
other non-perturbative phenomena. They also permit directly the connection to perturbation theory.
Employing functional approaches the infrared behavior of Landau gauge QCD vertex functions ex-
hibits dynamically induced scalar quark confinement. As can be analytically shown a linear rising
potential between heavy quarks is generated by infrared singularities in the dressed quark-gluon
vertex. The self-consistent mechanism that generates these singularities implies the existence of
scalar Dirac amplitudes of the full vertex and the quark propagator. These amplitudes can only be
present when chiral symmetry is broken, either explicitly or dynamically. The corresponding relations
thus constitute a novel mechanism that directly links $\chi_{\text{SB}}$ with confinement.

Both sets of scenarios are not yet complete, and the discussion on various aspects of them con-
tinues [218, 439, 440]. Nonetheless, the most developed scenarios of both cases, the com-
bined vortex and monopole scenario [438, 218] and the scenarios of Gribov and Zwanziger and of
Kugo and Ojima (GZKO) [438, 404, 439], provide already a quite encompassing view of the long-
distance shape of QCD. Based on both views, various scenarios for confinement, $\chi_{\text{SB}}$ and other
non-perturbative effects have been developed. However, if both views are correct then they can only
be different aspects of the same underlying physics, and it must be possible to relate them. [441]
discusses the current status of the understanding of this connection using smeared and cooled
configurations in lattice gauge theory to determine the properties of Green’s functions in the low-
momentum regime. It is found that the qualitative properties are essentially unchanged compared to
results on un-smeared configurations.

This is also the case when the configurations are smeared sufficiently strongly to reach the al-
most (anti-)self-dual domain. The fact that the low-momentum properties of the propagators keep
their qualitative properties under smearing provides two insights. On the one hand, this implies that
the long-range structure is shaping the low-momentum properties of propagators. This has already
been anticipated based on the findings in center-trivial groups [442, 443]. On the other hand, re-
producing the low-momentum properties of the Green’s functions in functional or other continuum
calculations indicates that at least part of the dynamics due to topological configurations has been
captured. Note that these findings do not reach far enough into the infrared to make (yet) a state-
ment on the asymptotic properties of the propagators in smeared (topological) field configurations.
Nonetheless, the fact that smearing does not qualitatively modify the low-momentum properties of
Green’s functions is in agreement with the expectations [444].
5 Conclusions

We reviewed the most prominent non-perturbative features of QCD, in particular the various aspects of chiral symmetry breaking ($\chi$SB): i) the dynamical $\chi$SB that leads to the pions being light pseudo-Goldstone bosons; ii) the anomaly, which eliminates the flavor-singlet axial $U(1)$ symmetry and prevents the $\eta'$-meson with a mass of order $\Lambda_{QCD}$ to be a Goldstone boson; iii) the explicit symmetry breaking from the quark masses, responsible for the pseudo-scalar mesons not being exactly massless. We presented numerical evidence for $\chi$SB and discussed its restoration at finite temperature and density. The understanding of the mechanisms goes well beyond perturbation theory and a rigorous treatment of them is presently only possible in the lattice regularization. We need aspects of the Dirac spectrum that rely on gauge fields of non-trivial topology. In particular the low-lying Dirac modes were analyzed by different groups in order to find ways to link $\chi$SB with quark confinement. Dimensionality and localization properties of these modes show that the non-perturbative vacuum can be characterized by various kinds of topological gauge field excitations. The Atiyah-Singer index theorem connects the total topological charge of gluonic field configurations with the number of zero modes, which induce the $U(1)_{A}$-anomaly. Interacting lumps of topological charge lead to low-lying Dirac modes which via the Banks-Casher relation determine the strength of spontaneous $\chi$SB. This is a “kinematical” scenario for spontaneous $\chi$SB. Let us try to conjecture a “dynamical” picture: The low momentum modes of quark fields change chirality, when they enter a combination of parallel color electric and magnetic fields, present in regions of non-vanishing topological charge density. Such fields force slow color charges into spiraling paths changing their momentum and conserving their spin. Fast moving charges are less influenced by such field combinations. This could explain the importance of low-lying Dirac modes for spontaneous $\chi$SB and clarify why Goldstone bosons do not survive the removal of low-lying Dirac modes and heavy hadrons with increasing removal tend to increase their masses. Hence field configurations with lumps of topological charge increase the density of low-lying Dirac eigenmodes with pronounced local chiral properties producing a finite chiral condensate.

As described above there are many unsolved interesting problems concerning the vacuum structure of QCD, confinement and $\chi$SB. Some of the most interesting questions for future work to generate progress in this field are as follows:

- Do chirally polarized low-energy modes condense? What is the physical origin of the band width $\Lambda_{ch}$ of condensing modes?
- Does the result that fermionic zero modes and chirality are localized on structures with fractal dimension $D = 2 - 3$ favor the vortex/domain-wall nature of the localization?
- Which kind of effective quark-gluon interactions are generated by dynamical $\chi$SB? Will this include a scalar confining force?
- Why do Goldstone bosons not survive the removal of low-lying Dirac modes?
- What is the relative contribution of the various interacting topological objects to the Dirac operator’s density of modes around zero virtuality?
- Do low-momentum modes change chirality in regions of non-vanishing topological charge density with electric and magnetic fields present and thus dynamically break chiral symmetry?
- Can one construct an explicit quantum state responsible for a dissipation-free flow of an electric current along an external magnetic field (chiral magnetic effect)?

An answer to these questions may hold the key to understand infrared QCD and the related phenomena, most prominently, confinement and dynamical $\chi$SB.
Acknowledgments

We thank Dmitry Antonov, Gerhard Ecker, Michael Engelhardt, Jeff Greensite, Urs M. Heller, Derek Leinweber and Štefan Olejník for helpful discussions. We thank Armand Fässler for choosing a very experienced and precise referee. We especially thank the careful referee who suggested many improvements. This research was supported by an Erwin Schrödinger Fellowship of the Austrian Science Fund FWF (“Fonds zur Förderung der wissenschaftlichen Forschung”) under Contract No. J3425-N27 (R.H.).

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